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Commutative Hopf Algebras, Lie Coalgebras, and Divided Powers

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This paper studies commutative bialgebras (graded or ungraded, over a field k of arbitrary characteristic) and their relationship to Lie coalgebras. In particular, if A is a commutative bialgebra then the vector space $QA = A^+/(A^+)^2$ (where A^+ is the augmentation ideal Ker ε of A) inherits from A the structure of a Lie coalgebra. One of our aims is to consider what can be said about A if the Lie coalgebra QA is given. This involves a new treatment of Lie coalgebras K and their (universal) coenveloping coalgebras (denoted here UcK); the latter have the structure of commutative Hopf algebras. (Lie coalgebras have previously been considered in [An, An2, Mi, Ni]; in the finite-dimensional case, Lie coalgebras and Lie algebras are just dual spaces of each other.)

At prime characteristic we make an additional assumption on A, namely, that A be a Γ-bialgebra (also known as divided power bialgebra or bialgebra with divided powers). A Γ-bialgebra, by definition, is a bialgebra A such that, first, the underlying algebra of A has the structure of a Γ-algebra, that is, is commutative and has a sequence $\{\gamma_i\}_{i\geqslant 0}$ of operators with properties like those which the operators $x\mapsto x^i/i!$ have at characteristic 0 (the precise definition is given in (3.1) below), and second, the coalgebra operations are compatible with the operators γ_i. (At characteristic 0, Γ-bialgebra = commutative bialgebra.) Γ-bialgebras A have been studied [An, An2, GL, Sc, Sj] in the case in which $A=\sum_{i\geqslant 0} A_i$ is graded and connected (that is, $A_0 = k$), and have applications to algebraic topology (homology of Eilenberg–MacLane spaces) and commutative algebra (Tor of a local ring). The principal result on connected graded Γ-bialgebras, due to André [An2], says (in our notation) that a connected graded Γ-bialgebra A is isomorphic to the coenveloping coalgebra UcK of

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the Lie coalgebra $K = Q_\Gamma A$ obtained by taking the augmentation ideal of A modulo its divided power square. This generalizes a characteristic 0 result of Milnor and Moore [MM], and is known as the Milnor–Moore–André theorem (a shorter proof has been given by Sjödin [Sj] for the special case where each A_i is finite dimensional, in which case the result can be rephrased in terms of Lie algebras and the duals of the enveloping algebras).

Because of the hypothesis that $A_0 = k$, this result gives no information in the ungraded case. Ungraded Γ-algebras have been studied in [Ro, Ro2, Be, BO]. In this paper we shall amalgamate the definition of Γ-algebra in the graded and ungraded cases and introduce the study of Γ-bialgebras and Hopf Γ-algebras (these being, respectively, comonoids and cogroups in the category of Γ-algebras) in the nonconnected graded case (and thus in particular in the ungraded case). As we shall show in this paper, there is a fine set of examples; indeed, the Hopf algebra $(UL)^0$ of representative functions on the enveloping algebra of a Lie algebra L, and more generally, the coenveloping coalgebra UcK of a Lie coalgebra K, is a Hopf Γ-algebra, as is also its irreducible component of the identity.

In Section 1 we shall give some preliminaries on cofree coalgebras, in particular on the cofree graded coalgebra TcV of a graded vector space V. In Section 2 we shall study TcV as a Hopf algebra, and shall show (Theorem 2.5) that the multiplication in TcV is obtained by taking extended shuffle products. In Section 3 we shall show (Theorem 3.1) that TcV is a Hopf Γ-algebra and has an appropriate universal property for Γ-bialgebras. In Section 4 we shall study Lie coalgebras, taking as our definition the property of having a covering by an (associative) coalgebra. (This is dual to the definition of Lie algebras by embeddings in associative algebras [MM]. This way of defining Lie coalgebras corresponds to that used in [An2, Ni], and differs from that by identities used in [Mi]; as Michaelis has shown, Lie coalgebra in our sense is the same as locally finite Lie coalgebra in his sense.) We shall give in Lemma 4.3 an explicit description of the elements in the coenveloping coalgebra UcK of a Lie coalgebra K, and shall use this to give a strikingly brief proof of the dual Poincaré–Birkhoff–Witt theorem. This result (Theorem 4.9) says that if we take the (descending) Lie filtration (defined in Section 4) on UcK, then the corresponding graded object Gr_LUcK is canonically isomorphic, as Hopf Γ-algebra, to BK (also denoted ΓK), the free Γ-algebra on (the underlying vector space of) K. This result was known in two cases (each with a long proof), namely, in the connected (that is, $K_0 = 0$) graded case [An2], and in the ungraded characteristic 0 case [Mi, Mi2]. We remark that BK is also the cofree irreducible cocommutative coalgebra and can be realized as the symmetric elements in the shuffle algebra of K; at characteristic 0 it is isomorphic to the symmetric algebra of K.

In Section 5, we shall show (Theorem 5.3) that if A is a Γ-bialgebra, $\{\gamma^i A\}_{i\geqslant 0}$ its descending filtration of Γ-powers of the augmentation ideal, and $\gamma^1 A/\gamma^2 A = K = Q_\Gamma A$ the associated Lie coalgebra, then the canonical map ζ of A to $\mathrm{Uc}K$ (which is a Γ-bialgebra map by Sections 3 and 4) has kernel $\bigcap_{i>0}\gamma^i A$ and image which is dense in the topology determined by the Lie filtration on $\mathrm{Uc}K$. In particular, ζ induces a Hopf Γ-algebra isomorphism $\mathrm{Gr}_\gamma A \cong \mathrm{Gr}_L \mathrm{Uc}K$, where $\mathrm{Gr}_\gamma A$ denotes the graded Γ-bialgebra associated to the above filtration on A; thus, by the dual PBW theorem, $\mathrm{Gr}_\gamma A \cong BK$, canonically, as Hopf Γ-algebras. This theorem has as an immediate consequence (Corollary 5.4) the Milnor–Moore–André theorem stated above, thus giving a new proof and generalization of that theorem. We shall also examine (in Section 5) the Lie algebra $L = K^*$, embedding UL in A^0 and relating the corresponding evaluation map $\rho: A \to (UL)^0$ to ζ.

We shall leave to an appendix the case in which the characteristic is 2 and there are (nonzero) odd elements. (André [An2] in effect excluded consideration of this case, and Sjödin [Sj] handled it in his case by introducing the notion of adjusted Lie algebra.)

The results of the paper, as well as giving the $\mathbb{N}$-graded case, apply to the $\mathbb{Z}_2$-graded case; a more general hypothesis on the gradings could be formulated if desired. A sequel to the present paper will give a determination of the irreducible Γ-bialgebras.

1. The Cofree Coalgebra TcV

Throughout the paper k denotes a given field, and all vector spaces, algebras, etc., and all tensor products, are over k. All gradings are by $\mathbb{N}$ or by $\mathbb{Z}_2$. Also V denotes a vector space which after about one more page will be graded. (Of course the ungraded case is included by taking $V = V_0$.) $\overline{TV}$ denotes the completion of the tensor algebra $TV = \sum_{n\geqslant 0} T^n V$, that is, the algebra of all infinite formal sums $a = a^0 + a^1 + \cdots$ where $a^j \in T^j V$. We take the usual topology on $\overline{TV}$, with a base of neighborhoods of 0 consisting of the sets

$$F^n\overline{TV} = \{a \in \overline{TV} \mid a^j = 0,\ j = 0,..., n-1\},$$

and also take the induced subspace topology on any vector subspace of $\overline{TV}$ that we consider.

We let $(\mathrm{Tc}V, \pi)$ (where π is the canonical map $\pi: \mathrm{Tc}V \to V$) denote the cofree coalgebra on V. For $c \in \mathrm{Tc}V$ we write $c^0 = \varepsilon c$, and for each $n > 0$ we define (using the notation of [Sw] for comultiplication)

$$c^n = \sum_{(c)} \pi c_{(1)} \otimes \cdots \otimes \pi c_{(n)} \qquad (= (\otimes^n \pi)\, \Delta_{n-1} c).$$

Thus $c^n \in T^n V$. Under the realization of TcV in [BL] as $k[x]^0_V$, $c \in \mathrm{Tc}V$ corresponds to the linear function f on the polynomial algebra $k[x]$ with $f(x^n) = c^n$ [BL, Theorem 2 with $\varphi = \pi$]. Therefore if $c^n = 0$ for all $n \geqslant 0$ then $c = 0$. Thus we may identify each $c \in \mathrm{Tc}V$ with the corresponding $c^0 + c^1 + \cdots \in \overline{TV}$, and so have $\mathrm{Tc}V \subseteq \overline{TV}$.

From [BL] we have the following charaterization of elements TcV.

LEMMA 1.1. *Suppose* $a \in \overline{TV}$. *Then* $a \in \mathrm{Tc}V$ *if and only if there exists (a necessarily unique)* $\sum_i b_i \otimes c_i \in \overline{TV} \otimes \overline{TV}$ *such that* $a^{m+n} = \sum_i b_i^m \otimes c_i^n$ *for all* $m, n \geqslant 0$; *in this case* $\sum b_i \otimes c_i \in \mathrm{Tc}V \otimes \mathrm{Tc}V$, *and* $\Delta a = \sum b_i \otimes c_i$. *In particular, if* $a \in \mathrm{Tc}V$ *then*

$$a^{m+n} = \sum_{(a)} a_{(1)}{}^m \otimes a_{(2)}{}^n \qquad (\forall m, n \geqslant 0).$$

If D is a coalgebra and $\varphi: D \to V$ is a linear map, then $\bar{\varphi}$ denotes the (unique) corresponding coalgebra map of D to TcV such that $\pi\bar{\varphi} = \varphi$. By Theorem 2 of [BL] we have, for $d \in D$,

$$(\bar{\varphi}d)^0 = \varepsilon d, \qquad (\bar{\varphi}d)^n = \sum_{(d)} \varphi d_{(1)} \otimes \cdots \otimes \varphi d_{(n)} \qquad (n = 1, 2, ...). \quad (1.1)$$

We shall also need a criterion from [BL] for a subspace of TcV to be a subcoalgebra. Suppose $\{v_j \mid j \in J\}$ is a basis of V, and for $j \in J$ define $\lambda_j \in V^*$ by $\lambda_j v_i = \delta_{ij}$. Then operators R_j, L_j $(j \in J)$ on TcV are defined by

$$\begin{aligned} R_j\left(\sum f^i\right) &= \sum \left(\left(\otimes^i 1\right) \otimes \lambda_j\right) f^{i+1}, \\ L_j\left(\sum f^i\right) &= \sum \left(\lambda_j \otimes \left(\otimes^i 1\right)\right) f^{i+1} \end{aligned} \qquad (1.2)$$

(where 1 denotes the identity map, in this case on V). Then Corollary 7 of [BL] says the following.

LEMMA 1.2. *A subspace* D *of* TcV *is a subcoalgebra if and only if* $R_j D \cup L_j D \subseteq D$ *for all* $j \in J$.

Now suppose that V is graded (by which we mean graded by $\mathbb{N}$ or by $\mathbb{Z}_2$), and let $|V|$ denote the underlying ungraded vector space of V. We write

$$(T^n V)_i = \sum_{i_1 + \cdots + i_n = i} V_{i_1} \otimes \cdots \otimes V_{i_n},$$

$$(\mathrm{Tc}V)_i = \{c \in \mathrm{Tc}\,|V| \mid \forall n, c^n \in (T^n V)_i\}$$

(for subscripts in $\mathbb{N}$ or $\mathbb{Z}_2$, in accord with those for V). We let V^* denote (as in [MM]) the graded vector space $\bigoplus_n \operatorname{Hom}(V_n, k)$, regarded as a subspace of $\operatorname{Hom}(|V|, k) = |V|^*$.

LEMMA 1.3. *$\sum_i(\mathrm{Tc}V)_i$ is a subcoalgebra of* $\mathrm{Tc}\,|V|$, *with the given grading, is graded as a coalgebra, and together with the restriction to it of π (also denoted π) is a realization, denoted* $\mathrm{Tc}\,V$, *of the cofree graded coalgebra on V.*

Proof. Choose a basis $\{v_j \mid j \in J\}$ of V with each v_j homogeneous and consider the associated operators R_j, L_j $(j \in J)$ on $\mathrm{Tc}\,|V|$. If $f \in (\mathrm{Tc}V)_i$, $j \in J$ and v_j has degree r then

$$(R_j f)^n = ((\otimes^n 1) \otimes \lambda_j) f^{n+1} \in (T^n V)_{i-r}.$$

Therefore $\sum(\mathrm{Tc}V)_i$ is invariant under each R_j (and L_j) and so, by Lemma 1.2, is a subcoalgebra of $\mathrm{Tc}\,|V|$. Now if $f \in (\mathrm{Tc}V)_i$ then $\Delta f = \sum_s g_s \otimes h_s$ where each g_s and h_s is homogeneous. We have

$$\sum_s g_s^m h_s^n = f^{m+n} \in (T^{m+n} V)_i.$$

Therefore the equality remains true if the summation is restricted to those indices s for which $\deg g_s + \deg h_s = i$. But then, by Lemma 1.1, $\Delta f = \sum g_s \otimes h_s$ for the sum over this restricted set of indices, and so our subcoalgebra is graded as a coalgebra. It remains to prove the cofree property. If D is a graded coalgebra and $\varphi\colon D \to V$ is a graded linear map (that is, $\varphi D_i \subseteq V_i$ for all i) we must show that $\bar{\varphi}$ (which maps D to $\mathrm{Tc}\,|V|$) satisfies $\bar{\varphi} D_i \subseteq (\mathrm{Tc}V)_i$. But this follows from (1.1). Q.E.D.

Of course, if $V = V_0$ then $\mathrm{Tc}V = \mathrm{Tc}\,|V|$. On the other hand, when the grading is by $\mathbb{N}$, if $V_0 = 0$ and $f \in \mathrm{Tc}V$ then $f^n = 0$ for all sufficiently large n; more precisely, if $f \in (\mathrm{Tc}V)_n$ then $f^m = 0$ for all $m > n$.

The above proof extends as well to give a graded version $A_V^0 \subseteq A_{|V|}^0$ of the coalgebra A_V^0 constructed in [BL] (the case of $\mathrm{Tc}V$ is that for which $A = k[x]$). A graded version of the dual coalgebra A^0 of a graded algebra A (which we shall use in the parts of Sections 4 and 5 concerned with Lie algebras) can be obtained as the special case of A_V^0 for which $V = k$ is homogeneous of degree 1, or equivalently, as follows. Let A be an $\mathbb{N}$-graded algebra and let $|A|$ denote the underlying ungraded algebra. Then the ungraded coalgebra $|A|^0$ consists of all representative functions on $|A|$, that is, all $f \in |A|^*$ for which there exists $\sum_s g_s \otimes h_s \in |A|^* \otimes |A|^*$

(necessarily unique) such that $f(ab)=\sum_s(g_s a)(h_s b)$ for all $a, b \in A$. Let A^0 be the graded space

$$A^0 = \bigoplus_n (|A|^0 \cap A_n^*).$$

For any $f \in |A|^0$ and any i, let $f_i \in A^*$ be defined by $f_i a_j = \delta_{ij} f a_j$ $(a_j \in A_j)$. Suppose $f \in A_n^0$ and $\Delta f = \sum_s g_s \otimes h_s$ $(\in |A|^0 \otimes |A|^0)$. Then $f(ab)=\sum_s \sum_{i=1}^n (g_{s,i} a)(h_{s,n-i} b)$ for all homogeneous $a, b \in A$, and hence $\Delta f = \sum_s \sum_{i=1}^n g_{s,i} \otimes h_{s,n-i}$. Essentially the same argument shows that if $g \in |A|^0$ then each $g_i \in |A|^0$. In particular each $g_{s,i}$ and $h_{s,n-i}$ above is representative. Therefore A^0 is a subcoalgebra of $|A|^0$ and is a graded coalgebra. We also have $A^0 = A^* \cap |A|^0$. If A is trivially graded, the present A^0 coincides with the usual (ungraded) A^0. (Our A^0 generalizes the A^g of [Sw], which was defined only in the locally finite case.) It can be seen that the present graded $A^0 = \{f \in A^* \mid \operatorname{Ker} f \supseteq \text{a cofinite ideal}\}$, where by a cofinite ideal we mean an ideal of finite codimension which, when A is graded, is assumed to be a graded ideal.

If A is $\mathbb{Z}_2$-graded, similar remarks apply; in this case, the $\mathbb{Z}_2$-graded A^0 consists of all of $|A|^0$.

For a coalgebra C or augmented algebra A we shall use the notation C^+ (or A^+), as in [Sw], to denote $\operatorname{Ker} \varepsilon$.

2. Shuffles and TcV

From now on V denotes a vector space which is allowed to be graded (by $\mathbb{N}$ or $\mathbb{Z}_2$) unless otherwise stated, and correspondingly, in the graded case, unless otherwise stated, all linear maps will be graded (homogeneous of degree 0), all twist maps τ (and hence the definition of commutativity, tensor product of algebras, Hopf algebra, etc.) will obey the usual sign convention, and all algebras and coalgebras will be graded.

Let M, u, and S denote, respectively, the unique coalgebra map $M\colon \mathrm{Tc}V \otimes \mathrm{Tc}V \to \mathrm{Tc}V$, $u\colon k \to \mathrm{Tc}V$, and $S\colon (\mathrm{Tc}V)^{\mathrm{op}} \to \mathrm{Tc}V$ such that $\pi M = \pi \otimes \varepsilon + \varepsilon \otimes \pi$, $\pi u = 0$, and $\pi S = -\pi$, that is, $M = \overline{\pi \otimes \varepsilon + \varepsilon \otimes \pi}$, $u = \bar{0}$, and $S = \overline{-\pi}$. With these maps as multiplication, unit and antipode, TcV is a Hopf algebra (as is observed in the ungraded case in [Mi]). Indeed, to show that u is a unit map, it suffices to show that $\pi M(u \otimes 1) = \pi = \pi M(1 \otimes u)$; but $\pi M(u \otimes 1) = \pi u \otimes \varepsilon + \varepsilon u \otimes \pi = \varepsilon_k \otimes \pi = \pi$. Associativity similarly follows by a routine computation showing that $\pi M(M \otimes 1) = \pi M(1 \otimes M)$. Hence Tc$V$ is a bialgebra. To show that S is an antipode, note that if $c \in \mathrm{Tc}V$ then

$$\pi \sum_{(c)} c_{(1)} S c_{(2)} = \sum_{(c)} (\pi c_{(1)} \varepsilon S c_{(2)} + \varepsilon c_{(1)} \pi S c_{(2)}) = 0.$$

Also, if $d \in \mathrm{Tc}V$ then (since $\Delta S = (S \otimes S)\tau\Delta$)

$$\Delta \sum_{(d)} d_{(1)} S d_{(2)} = \sum_{(d)} d_{(1)} S d_{(4)} \otimes d_{(2)} S d_{(3)}.$$

Therefore

$$\left(\sum_{(d)} d_{(1)} S d_{(2)}\right)^{n+1} = \sum_{(d)} (d_{(1)} S d_{(4)})^n \otimes (d_{(2)} S d_{(3)})^1 = 0$$

for all $n \geqslant 0$. Also $\varepsilon \sum d_{(1)} S d_{(2)} = \varepsilon d$ and so $\sum d_{(1)} S d_{(2)} = u\varepsilon d$, and similarly for $\sum (S d_{(1)})\, d_{(2)}$, as desired.

The above bialgebra structure on $\mathrm{Tc}V$ can also be characterized by a useful universal property, as follows. Regard V as the augmentation ideal, with zero multiplication, of the corresponding augmented algebra $k + V$. Let π' denote the projection of $k + V$ onto V, and $\varepsilon + \pi$ the map of $\mathrm{Tc}V$ to $k + V$, $c \mapsto \varepsilon(c) + \pi(c)$. Then $\varepsilon + \pi$ is an augmented algebra map, and the pair $(\mathrm{Tc}V, \varepsilon + \pi)$ is the cofree bialgebra on the augmented algebra $k + V$, that is, if H is a bialgebra and $\sigma\colon H \to k + V$ is an augmented algebra map (note that giving σ is equivalent to giving a linear map of H^+ to V vanishing on H^{+2}) then there is a unique bialgebra map $\psi\colon H \to \mathrm{Tc}V$ (namely, $\psi = \overline{\pi'\sigma}$) such that $\sigma = (\varepsilon + \pi)\psi$ (or equivalently, such that $\pi'\sigma = \pi\psi$). Indeed $\psi = \overline{\pi'\sigma}$ preserves multiplication since $\pi\psi M_H = \pi'\sigma M_H = \pi' M_{k+V}(\sigma \otimes \sigma) = \pi'\sigma \otimes \varepsilon + \varepsilon \otimes \pi'\sigma = \pi M(\psi \otimes \psi)$, and ψ preserves the unit since $\pi\psi u_H = \pi'\sigma u_H = 0 = \pi u$.

We rephrase in the following statement what has just been proved above.

PROPOSITION 2.1. *$\mathrm{Tc}V$, with the operations given above, is a Hopf algebra, and if H is any bialgebra and $\varphi\colon H \to V$ a linear map such that $\varphi(k + (H^+)^2) = 0$ then $\bar{\varphi}\colon H \to \mathrm{Tc}V$ is a bialgebra map and is the unique such map ψ such that $\pi\psi = \varphi$.*

If H is also a Hopf algebra then $\bar{\varphi}$ also preserves antipode [Sw, p. 81] and $\varphi S_H = -\varphi$.

For $v \in V_0$ let e^v denote $1 + v + v \otimes v + \cdots \in \overline{TV}$. Also let $G(\mathrm{Tc}V)$ denote the multiplicative group of grouplike elements of $\mathrm{Tc}V$.

LEMMA 2.2. *The map $V_0 \to G(\mathrm{Tc}V)$, $v \mapsto e^v$, is an isomorphism of the additive group of V_0 to $G(\mathrm{Tc}V)$.*

Proof. It follows from Lemma 1.1 that $e^v \in G(\mathrm{Tc}V)$ for $v \in V_0$. If $g \in G(\mathrm{Tc}V)$ and $g^1 = v$ then $g^0 = \varepsilon g = 1$, $g^{m+1} = g^m \otimes g^1 = v \otimes \cdots \otimes v$ ($m + 1$ times) and so $g = e^v$. If V is graded by $\mathbb{N}$, $v \in V_0$ since $g \in \mathrm{Tc}V$. If V is graded by $\mathbb{Z}_2$, $v \in V_0$ is forced by making the evenness of grouplikes part of their definition. The map $v \mapsto e^v$ is obviously bijective. If $v, w \in V_0$ then

$\pi(e^v e^w) = \pi(e^v)\,\varepsilon(e^w) + \varepsilon(e^v)\,\pi(e^w) = v + w$, and since $e^v e^w$ is grouplike, $e^v e^w = e^{v+w}$. Q.E.D.

We now determine an explicit formula for the multiplication in $\mathrm{Tc}\,V$. We first consider $(\mathrm{Tc}\,V)^1$, the irreducible component of $1 = 1 + 0 + \cdots$ in $\mathrm{Tc}\,V$, which we shall show is (a realization of) the shuffle algebra $\mathrm{Sh}\,V$. In the process we give a short new treatment of $\mathrm{Sh}\,V$, the main feature being that we start with what we know to be a Hopf algebra, namely, $(\mathrm{Tc}\,V)^1$.

Note that the underlying coalgebra of $(\mathrm{Tc}\,V)^1$ (together with the restriction of π) is the cofree pointed irreducible coalgebra on V, that is, if D is a pointed irreducible coalgebra and $f\colon D^+ \to V$ is linear then there is a unique coalgebra map $\bar{f}\colon D \to (\mathrm{Tc}\,V)^1$ such that $\pi\bar{f}|_{D^+} = f$. This follows from the universal property of $\mathrm{Tc}\,V$ together with the facts that the homomorphic image of a pointed irreducible coalgebra is pointed irreducible and that 1 is the unique grouplike g of $\mathrm{Tc}\,V$ with $\pi g = 0$.

Moreover, under the identification of $\mathrm{Tc}\,V$ with a subspace of $\overline{TV}$, $(\mathrm{Tc}\,V)^1$ coincides with the subspace TV. To see this, note first that if $v_1,\dots, v_n \in V$ and $0 \leqslant j \leqslant m$ we have

$$(v_1 \otimes \cdots \otimes v_n)^m = \sum_{i=0}^{n} (v_1 \otimes \cdots \otimes v_i)^j \otimes (v_{i+1} \otimes \cdots \otimes v_n)^{m-j}.$$

Hence, by Lemma 1.1,

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=0}^{n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n), \qquad (2.1)$$

and so TV (with this comultiplication) is a subcoalgebra of $\mathrm{Tc}\,V$. Now write $C_n = \sum_{i=0}^{n} T^i V$. We prove by induction that $\Lambda^n k = C_{n-1}\Lambda$ for $n \geqslant 1$. Suppose it holds for n. Recalling [Sw, p. 179] that $\Lambda^{n+1}k = (\Lambda^n k)\Lambda k = \{f \in \mathrm{Tc}\,V \mid \Delta f \in \Lambda^n k \otimes \mathrm{Tc}\,V + \mathrm{Tc}\,V \otimes k\}$, we see by (2.1) that $C_n \subseteq \Lambda^{n+1}k$. On the other hand, if $\Delta f \in C_{n-1} \otimes \mathrm{Tc}\,V + \mathrm{Tc}\,V \otimes k$ then, for $i > 0$,

$$f^{n+i} = \sum f_{(1)}{}^n f_{(2)}{}^i = 0,$$

so that $f \in C_n$ as desired. Since $(\mathrm{Tc}\,V)^1 = \bigcup_n \Lambda^n k$, we have $(\mathrm{Tc}\,V)^1 = TV$.

The multiplication in $(\mathrm{Tc}\,V)^1$ is the shuffle multiplication, that is, we have (using the vector space identification of $(\mathrm{Tc}\,V)^1$ with TV)

$$M((v_1 \otimes \cdots \otimes v_r) \otimes (v_{r+1} \otimes \cdots \otimes v_n)) = \sum_{\psi} \psi(v_1 \otimes \cdots \otimes v_n) \qquad (2.2)$$

where ψ ranges over all $(r, n-r)$-shuffles (that is, permutations of $1,\dots, n$ preserving the order of $1,\dots, r$ and of $r+1,\dots, n$) and, for any permutation ψ,

$$\psi(v_1 \otimes \cdots \otimes v_n) = v_{1\psi} \otimes \cdots \otimes v_{n\psi}, \qquad i\psi = \psi^{-1} i$$

(with, in the graded case, $v_1, \ldots, v_n$ homogeneous and the appropriate power of -1 inserted). That the multiplication M in $(\mathrm{Tc}V)^1$ is indeed given by this formula may be seen (exercise) by expanding

$$\Delta_{j-1}((v_1 \otimes \cdots \otimes v_r) \otimes (v_{r+1} \otimes \cdots \otimes v_n)),$$

applying $\otimes^j(\pi \otimes \varepsilon + \varepsilon \otimes \pi)$ to the resulting expression, and using (1.1).

It follows from Proposition 2.1 that $(\mathrm{Tc}V)^1$ is the cofree irreducible Hopf algebra on the augmented algebra $k + V$, where $V^2 = 0$, and so coincides with what Newman and Radford [NR] denote by $\mathrm{CH}(V)$, for the case $V^2 = 0$ (V ungraded). The other cases of $\mathrm{CH}(V)$ considered in [NR] can be fit into our framework by replacing $\pi \otimes \varepsilon + \varepsilon \otimes \pi$ in the definition of M by $\pi \otimes \varepsilon + \varepsilon \otimes \pi + \mu(\pi \otimes \pi)$ where μ is a multiplication map $V \otimes V \to V$ making V into an associative (but not necessarily unital) algebra V_μ. We thus obtain a bialgebra $\mathrm{Tc}V_\mu$, and $(\mathrm{Tc}V_\mu)^1 = \mathrm{CH}(V_\mu)$. It can be shown that $\mathrm{Tc}V_\mu$ has an antipode if and only if V_μ equals its Jacobson radical.

We now determine a formula for the multiplication in all $\mathrm{Tc}V$.

LEMMA 2.3. *If $a \in \mathrm{Tc}V$ then Δa, regarded as an element of $\overline{TV \otimes TV}$, equals $\sum_{n \geqslant 0} \Delta(a^n)$ (where $\Delta(a^n)$ is given by* (2.1)); *that is,*

$$\sum_{(a), i \leqslant n} a_{(1)}{}^i \otimes a_{(2)}{}^{n-i} = \Delta(a^n) \qquad (n \geqslant 0).$$

Proof. Let ρ be the linear map

$$\rho \colon \overline{TV} \otimes \overline{TV} \to (TV)^{\mathbb{N} \times \mathbb{N}},$$

$$\left(\rho\left(\sum b_i\right) \otimes \left(\sum c_j\right)\right)(l, m) = b_l \otimes c_m \in T^{l+m}V.$$

(This corresponds to a special case of the map ρ of [BL].) We have

$$\left(\rho \sum_{(a), i+j=n} a_{(1)}{}^i \otimes a_{(2)}{}^j\right)(l, m)$$
$$= \delta_{l+m,n} \sum_{(a)} a_{(1)}{}^l \otimes a_{(2)}{}^m = \delta_{l+m,n} a^n.$$

Also, with $b = a^n$, we have

$$(\rho(\Delta b))(l, m) = \sum_{(b)} b_{(1)}{}^l \otimes b_{(2)}{}^m = b^{l+m} = \delta_{l+m,n} a^n.$$

Since ρ is injective, as noted in [BL], this gives the result. Q.E.D.

COROLLARY 2.4. *If $a \in \mathrm{Tc}V$, $m > 1$, and $n \geqslant 0$ then*

$$\sum_{(a), i_1 + \cdots + i_m = n} a_{(1)}{}^{i_1} \otimes \cdots \otimes a_{(m)}{}^{i_m} = \Delta_{m-1}(a^n).$$

Let $\overline{\mathrm{Sh}V}$ denote the completed shuffle product, that is, $\overline{\mathrm{Sh}V} = \overline{TV}$ as a vector space, with multiplication given by $(\sum a^i)(\sum b^j) = \sum_n \sum_{i=0}^n a^i b^{n-i}$ where here juxtaposition denotes the shuffle product. We now show that this also gives the multiplication in $\mathrm{Tc}V$.

THEOREM 2.5. *The multiplication $M = \overline{\pi \otimes \varepsilon + \varepsilon \otimes \pi}$ in $\mathrm{Tc}V$ coincides with that arising from $\overline{\mathrm{Sh}V}$, that is, as an algebra $\mathrm{Tc}V$ is a subalgebra of $\overline{\mathrm{Sh}V}$.*

Proof. Suppose $a, b \in \mathrm{Tc}V$. By (1.1)

$$(ab)^n = \otimes^n (\pi \otimes \varepsilon + \varepsilon \otimes \pi) \sum_{(a),(b)} (a_{(1)} \otimes b_{(1)}) \otimes \cdots \otimes (a_{(n)} \otimes b_{(n)}).$$

This expression depends only on

$$\sum_{(a), 0 \leqslant i_1, \ldots, i_n \leqslant 1} a_{(1)}^{i_1} \otimes \cdots \otimes a_{(n)}^{i_n}$$

and the corresponding expression involving b. By Corollary 2.4, this expression involving a is a sum of certain components of $\Delta_{n-1} a^0, \ldots, \Delta_{n-1} a^n$, and similarly for b. Hence we do not change $(ab)^n$ if we replace a by $\sum_{i \leqslant n} a^i$ and b by $\sum_{j \leqslant n} b^j$. Q.E.D.

We remark that it follows that the multiplication in $\mathrm{Tc}V$ coincides with the convolution product in $\mathrm{Hom}(k[x], \mathrm{Sh}V)$ where, as in [BL], $\mathrm{Tc}V$ is identified with a subspace of $\mathrm{Hom}(k[x], TV)$, but now we are regarding the vector space TV as the algebra $\mathrm{Sh}V$ and the vector space $k[x]$ as the divided power coalgebra (that is, $\Delta x^n = \sum_{i=0}^n x^i \otimes x^{n-i}$).

We now determine a formula for the antipode S in $\mathrm{Tc}V$. Let τ' denote the linear map on TV for which, for all n and $v_1, \ldots, v_n \in V$, $\tau'(v_1 \otimes \cdots \otimes v_n) = v_n \otimes \cdots \otimes v_1$ (with the appropriate modification if V is graded).

PROPOSITION 2.6. *For every n and $f \in \mathrm{Tc}V$,*

$$(Sf)^n = (-1)^n \tau'(f^n).$$

Proof. Since $S = \overline{-\pi}$ where $-\pi$ is regarded as a linear map from $(\mathrm{Tc}V)^{\mathrm{op}}$ to V, it follows from (1.1) that

$$(Sf)^n = (-1)^n \tau'(\otimes^n \pi) \Delta_{n-1} f.$$

It follows from Corollary 2.4 that the right-hand side of this depends only on f^n. But if $b = v_1 \otimes \cdots \otimes v_n$ then, by (2.1), $(\otimes^n \pi) \Delta_{n-1} b = v_1 \otimes \cdots \otimes v_n$. Q.E.D.

One of the ways in which shuffle multiplication arises in coalgebra theory is shown in the next lemma. Suppose D is a coalgebra containing the grouplike element g, and decompose D as $kg + D^+$. Let $\psi: k \to D$, $\alpha \mapsto \alpha g$, and $\varphi = 1 - \psi\varepsilon: D \to D$. For $n \geq 0$ and $x \in D$, define $\Delta_n^+ x$ to be the component of $\Delta_n x$ in $D^+ \otimes \cdots \otimes D^+$, that is, $\Delta_n^+ = \Delta_n^+ \varphi = (\otimes^{n+1} \varphi) \Delta_n$. Thus $\Delta_n^+ = (1 \otimes \cdots \otimes 1 \otimes \Delta^+) \Delta_{n-1}^+$, the comultiplication Δ^+ $(= \Delta_1^+)$ on D^+ is associative, and for $x \in D^+$, $\Delta_0^+ = x$ and $\Delta_1^+ x = \Delta x - g \otimes x - x \otimes g$. We write $g^{[n]}$ for $g \otimes \cdots \otimes g$ (n g's).

LEMMA 2.7. *For $n \geq 0$ and $x \in D$,*

$$\Delta_n x = (g^{[n+1]} \cdot \varepsilon(x)) + \sum_{i=0}^{n} g^{[n-i]} \cdot \Delta_i^+ x$$

where $\cdot$ denotes the shuffle multiplication in TD.

Proof. Suppose the formula holds for n (this being true if $n = 0$) and apply $(\otimes^n 1) \otimes \Delta$. Since $\Delta g = g \otimes g$ and $\Delta y = g \otimes y + y \otimes g + \Delta^+ y$ for $y \in D^+$, the term $g^{[n-i]} \cdot \Delta_i^+ x$ goes to the sum of those terms of $g^{[n-i+1]} \cdot \Delta_i^+ x$ with one or two g's in the last two positions plus the sum of those terms of $g^{[n-i]} \cdot \Delta_{i+1}^+ x$ with no g in the last two positions. Summed over i, this gives the formula for $n+1$. Q.E.D.

The formula may also be written as

$$\Delta_n x = ((1 + g + g^{[2]} + \cdots) \cdot (\varepsilon(x) + \Delta_0^+ x + \Delta_1^+ x + \cdots))^{n+1},$$

that is, $\bar{1} x = e^g \cdot \bar{\varphi} = \overline{\psi\varepsilon g} \cdot \overline{1 - \psi\varepsilon x}$ for all $x \in D$. This may be seen to be equivalent to $\bar{1} = \overline{\psi\varepsilon} * \overline{1 - \psi\varepsilon}$ (the convolution product in $\mathrm{Hom}(D, \mathrm{Tc}D)$).

3. DIVIDED POWERS

We show next that $\mathrm{Tc}V$ is a Hopf algebra with divided powers, and has a universal mapping property analogous to that in Proposition 2.1. We first modify the definition of a Γ-algebra as given in the graded case in [Ca, GL, An2] to extend the definition in the ungraded case used in [Ro, Ro2, Be, BO].

Let A be a commutative algebra (which at characteristic 2 is assumed to be strictly commutative, that is, $a^2 = 0$ for all odd $a \in A$). Also let $I_0 = I_0(A)$ be a proper ideal of A_0, and $I = I(A) = I_0 + \sum_{n>0} A_{2n}$ $(= I_0$ if the grading is by $\mathbb{Z}_2$). By a system of divided powers on I we mean a family $\gamma_0, \gamma_1, \ldots$ of mappings which attaches to each $x \in I$ a sequence $\gamma_0 x, \gamma_1 x, \ldots$ of elements of A satisfying

$$\begin{gathered} \gamma_0(x) = 1, \qquad \gamma_1 x = x, \\ \gamma_n x \in A_{mn} \quad \text{if } x \in A_m, \qquad \gamma_j I_0 \subseteq I_0 \quad \text{if } j > 0; \end{gathered} \tag{3.1.1}$$

$$(\gamma_m x)(\gamma_n x) = \binom{m+n}{m} \gamma_{m+n} x; \tag{3.1.2}$$

$$\gamma_n(x+y) = \sum_{i+j=n} \gamma_i(x)\, \gamma_j(y); \tag{3.1.3}$$

$$\gamma_n(xy) = 0 \qquad \text{if } x, y \text{ are homogeneous of odd degree and } n \geqslant 2; \tag{3.1.4}$$

$$\gamma_n(xy) = x^n \gamma_n y \qquad \text{if } x, y \text{ are homogeneous of even degree and } y \in I; \tag{3.1.5}$$

$$\gamma_m(\gamma_n(x)) = \frac{(mn)!}{m!(n!)^m} \gamma_{mn} x \qquad \left(= \left(\prod_{i=1}^{m-1} \binom{in+n-1}{n-1} \right) \gamma_{mn} x \right). \tag{3.1.6}$$

We then say that A (or more precisely (A, I_0) or (A, I_0, γ)) is a divided power algebra, or Γ-algebra. The notation $x^{[i]}$ will also be used for $\gamma_i(x)$. By taking $I_0 = 0$ (resp. $A = A_0$) we essentially get as a special case the notion of $\mathbb{N}$-graded (resp. ungraded) Γ-algebra studied by Cartan [Ca, Exposé 7] and André [An2] (resp. Roby [Ro, Ro2], Berthelot [Be], and Berthelot and Ogus [BO, Chap. 3]). (In [Ro, Ro2, Be, BO] the ground ring was not necessarily a field. In [Ca], axiom (3.1.6) was not actually made part of the definition, but was shown to hold under suitable hypotheses, and was sometimes assumed; in [An2], axiom (3.1.6) was not mentioned, but it appears that implicit use of it was made at one point in the proof in [An2], as will be noted below. Sjödin [Sj], who refers to the definition of Γ-algebra in [An2], also makes implicit use of (3.1.6) in using the freeness of the free Γ-algebra. In [Ca, An2] the definition was modified at characteristic 2 to require that the γ_n be defined on all A_m with $m > 0$ or equivalently that $A = 0$ for odd m; we are able to avoid this restriction (as did Sjödin [Sj] in his case) with the assumption that the multiplication is *strictly* commutative (see the Appendix).)

The name "divided power" is merited since, by axioms (3.1.1) and (3.1.2), $n!\, \gamma_n(x) = x^n$ for all $n \geqslant 0$. At characteristic 0, (A, I_0) always has a (unique) Γ-structure, namely, $\gamma_n(x) = x^n/n!$. At characteristic p, all γ_k are determined by γ_p.

A (graded) algebra map $\varphi\colon A \to B$ of Γ-algebras (A, I_0), (B, J_0) is called a

Γ-algebra map if $\varphi I_0 \subseteq J_0$ and $\varphi(a^{[i]}) = (\varphi a)^{[i]}$ for all i and all $a \in I(A)$. The tensor product of Γ-algebras (A, I_0) and (B, J_0) is given by $(A \otimes B, I_0 \otimes B_0 + A_0 \otimes J_0)$ with its canonical Γ-structure determined by using the relations $x \otimes y = (x \otimes 1)(1 \otimes y) = (1 \otimes y)(x \otimes 1)$ together with (3.1.3)–(3.1.5); the canonical maps $A \to A \otimes B$ and $B \to A \otimes B$ are Γ-maps (the proof of the existence of this Γ-structure in [Ro] makes use of an extra assumption on I_0 and J_0, namely, that they have complementary subalgebras; however, in our case, since we are assuming that the base ring is a field, the existence proof in [Ca] goes through). We take as the Γ-structure on the base field k ($= k_0$) the trivial Γ-structure, with $I_0 = 0$.

With the above coproduct (namely, tensor product) and initial object (namely, k) in the category of Γ-algebras, we may define the categories of Γ-algebra comonoids and Γ-algebra cogroups, the objects of which we call, respectively, Γ-bialgebras (or divided power bialgebras) and Hopf Γ-algebras (or Hopf divided power algebras). Thus a Γ-bialgebra (resp. Hopf Γ-algebra) is defined to be a Γ-algebra (A, I_0) which is simultaneously a bialgebra (resp. Hopf algebra) such that Δ and ε (resp. Δ, ε, and S) are Γ-maps (in particular, I_0 is a bi-ideal, resp. Hopf ideal; given that I_0 is a bi-ideal, ε is automatically a Γ-map). In [An2, Sj] $\mathbb{N}$-graded Γ-bialgebras for which $I_0 = 0$ were studied (essentially just for the connected case $A_0 = k$). It appears that the ungraded case (or the graded case with $I_0 \neq 0$) has not been previously studied.

We now recall the Γ-structure on $\operatorname{Sh} V$, combining the cases considered by Roby [Ro2] (V ungraded, $I_0 = V + V \otimes V + \cdots$) and André [An2] ($V$ graded, $I_0 = 0$). Let V be graded. In the Hopf algebra $A = (\operatorname{Tc} V)^1 = \operatorname{Sh} V$ we take

$$I_0 = A_0^+ = V_0 + V_0 \otimes V_0 + \cdots$$

(thus $A_0 = k + I_0$, and I_0 is a Hopf ideal of A_0). Then A has a Γ-structure, determined by

$$\gamma_n(v_1 \otimes \cdots \otimes v_j) = \sum \sigma(\otimes^n (v_1 \otimes \cdots \otimes v_j)) \tag{3.2}$$

(for homogeneous $v_1, ..., v_j \in V$ such that $v_1 \otimes \cdots \otimes v_j \in I$) where the summation is over all $(j, ..., j)$-shuffles σ for which $\sigma(j) < \sigma(2j) < \cdots < \sigma(nj)$. (These shuffles give a set of representatives for the left cosets in the symmetric group of the subgroup of permutations of $\{1, ..., nj\}$ which permute the n blocks of length j and the elements within them. In particular, $\gamma_n v = \otimes^n v$.) A proof that Δ is a Γ-map is indicated in [An2, pp. 26–27].

If A is a Γ-algebra we denote the divided power square of A^+ by $\gamma^2 A$, that is,

$$\gamma^2 A = A^+ A^+ + \text{span of } \{\gamma_n x \mid x \in I(A), n \geqslant 2\}.$$

THEOREM 3.1. *There exists a unique Γ-structure on* $(\mathrm{Tc}V, (\mathrm{Tc}V)_0^+)$ *which agrees on* $\mathrm{Sh}V$ *with the Γ-structure given above and is such that each γ_n is continuous. With this Γ-structure,* $\mathrm{Tc}V$ *is a Hopf Γ-algebra and has the following universal property: if H is a Γ-bialgebra and $\varphi\colon H \to V$ a linear map such that $\varphi(k+\gamma^2 H)=0$ then the bialgebra map $\bar{\varphi}$ is also a Γ-map.*

Proof. Suppose $a=a^1+\cdots \in (\mathrm{Tc}V)_j^+$ where j is even. For $s \geqslant 1$ write $\xi_s a = a^1 + \cdots + a^s$. Then for the sequence of divided powers on $\mathrm{Sh}V$ described above we have, for $n \geqslant 1$,

$$\gamma_n(\xi_s a) = \sum_{m_1+\cdots+m_s=n} (\gamma_{m_1} a^1)\cdots(\gamma_{m_s} a^s).$$

The summand displayed on the right side is in $(T^l V)_{nj}$ where $l=m_1+2m_2+\cdots+sm_s$. Therefore replacement of s by any larger integer does not affect terms contained in $T^0V+\cdots+T^sV$. Hence $\lim_{s\to\infty}\gamma_n(\xi_s a)$ is defined (in $\overline{TV}$); we set $\gamma_n a$ equal to this limit, a definition which is forced by the conditions of agreement on $\mathrm{Sh}V$ and continuity, thus giving uniqueness.

We now show that $\gamma_n a \in \mathrm{Tc}V$. Let $\Delta a = \sum_{i=1}^r b_i \otimes c_i$; we may assume for each i that b_i is homogeneous, $b_i=1$, or $b_i \in \operatorname{Ker}\varepsilon$ and similarly for c_i, and (since $\varepsilon a=0$) that either $b_i \neq 1$ or $c_i \neq 1$. In what follows, if $c_i=1$ we replace $\gamma_{m_i} c_i$ by $c_i^{m_i}$ and $b_i^{m_i}$ by $\gamma_{m_i} b_i$. We claim, for all $s, t \geqslant 0$ and $n>1$, that

$$(\gamma_n a)^{s+t} = \sum_{m_1+\cdots+m_r=n} (b_1^{m_1}\cdots b_r^{m_r})^s \otimes (\gamma_{m_1}c_1\cdots\gamma_{m_r}c_r)^t;$$

here any summand containing an undefined factor $\gamma_{m_i}c_i$ with $m_i \geqslant 2$ and c_i (and hence also b_i) of odd degree is regarded as being 0 (which is in accord with (3.1.4)). It suffices to show this with each a, b_i, c_i replaced, respectively, by $\xi_{s+t}a$, $\xi_{s+t}b_i$, $\xi_{s+t}c_i$, and so it suffices (by Lemma 1.1) to show that

$$(\Delta\gamma_n\xi_{s+t}a)^{s\otimes t} = \left(\sum_{m_1+\cdots+m_r=n} (\xi_{s+t}b_1)^{m_1}\cdots(\xi_{s+t}b_r)^{m_r} \otimes \gamma_{m_1}\xi_{s+t}c_1\cdots\gamma_{m_r}\xi_{s+t}c_r\right)^{s\otimes t}$$

where the superscript $s\otimes t$ denotes the component in $T^sV\otimes T^tV$. But

$$(\Delta\gamma_n\xi_{s+t}a)^{s\otimes t} = (\gamma_n\,\Delta\xi_{s+t}a)^{s\otimes t} = \left(\gamma_n \sum_{i=1}^r \xi_{s+t}b_i \otimes \xi_{s+t}c_i\right)^{s\otimes t}$$

which proves the claim. It then follows from Lemma 1.1 that $\gamma_n a \in \mathrm{Tc}V$. Properties (3.1.1)–(3.1.6) in $\mathrm{Tc}V$ are obtained from the corresponding properties in $\mathrm{Sh}V$ by restriction to partial sums, and similarly for $\gamma_n \Delta = \Delta \gamma_n$. In the same manner, that S is a Γ-map on $\mathrm{Tc}V$ follows from the fact that S is a Γ-map on $\mathrm{Sh}V$; the latter fact may be proved from the formulas for S (Proposition 2.6) and $\gamma_n(v_1 \otimes \cdots \otimes v_j)$ (see (3.2)).

It remains to prove, under the given hypotheses, that $\bar{\varphi}$ is a Γ-map. Since the tensor product of Γ-maps is a Γ-map, $1 \otimes \cdots \otimes 1 \otimes \Delta$ (where Δ denotes Δ_H) is a Γ-map and hence each Δ_j is a Γ-map. Now suppose $x \in I(H)$ with x homogeneous. By Lemma 2.7, for $s \geqslant 1$,

$$\Delta_{s-1} x = \sum_{i=1}^{s} 1^{[s-1]} \cdot \Delta_{i-1}{}^{+} x,$$

which when expanded we can write as $\sum_{j=1}^{r} x_{ji} \otimes \cdots \otimes x_{js}$ where for each j and i, x_{ji} is homogeneous and either $x_{ji} = 1$ or $x_{ji} \in H^{+}$. We have

$$(\bar{\varphi}\gamma_n x)^s = (\otimes^s \varphi)\, \Delta_{s-1} \gamma_n x = (\otimes^s \varphi)\, \gamma_n \Delta_{s-1} x$$

$$= \sum_{l_1 + \cdots + l_r = n} (\otimes^s \varphi) \prod_{i=1}^{r} \gamma_{l_i}(x_{i1} \otimes \cdots \otimes x_{is}).$$

Since $\varphi(k + \gamma^2 H^{+}) = 0$, the product on the right is annihilated by $\otimes^s \varphi$ unless the nonzero l_i are 1 and for each $j = 1,\ldots, s$ there is exactly one pair ij for which $l_i = 1$ and $x_{ij} \in H^{+}$. A factor with exactly t tensorands in H^{+} comes from a summand of $1^{[s-t]} \cdot \Delta_{t-1}{}^{+} x$. It follows that

$$(\bar{\varphi}\gamma_n x)^s = \sum (\otimes^s \varphi)\, (\Delta_{i_1 - 1}{}^{+} x \cdot \cdots \cdot \Delta_{i_n - 1}{}^{+} x)$$

$$= \sum (\bar{\varphi} x)^{i_1} \cdots (\bar{\varphi} x)^{i_s}$$

where the summations are over all $i_1,\ldots, i_n$ such that $i_1 + \cdots + i_n = s$ and $i_1 \leqslant \cdots \leqslant i_n$. Hence $\bar{\varphi}\gamma_n x$ depends only on $\bar{\varphi} x$ and n, but not on H or φ, and so remains the same if H is replaced by $\mathrm{Tc}V$, x by $\bar{\varphi} x$, and φ by π (and hence $\bar{\varphi}$ by 1). Therefore $\bar{\varphi}\gamma_n x = \gamma_n \bar{\varphi} x$. Q.E.D.

We shall need to consider bigraded vector spaces (and algebras, coalgebras, etc.). By a bigraded vector space we mean a vector space W such that $W = \bigoplus W_{ij}$ for subspaces W_{ij} where every $i \in \mathbb{N}$ and either every $j \in \mathbb{N}$ or every $j \in \mathbb{Z}_2$ (for $j \in \mathbb{N}$, our W_{ij} corresponds to $W_{i,j-i}$ in the notation of [MM]. By the second index grading of W we mean that obtained by setting $W_j = \sum_{i \geqslant 0} W_{ij}$. If V is graded we regard TV as being bigraded with $(TV)_{ij} = (T^i V)_j$. If V, W are bigraded with the same kind of second index grading (that is, both by $\mathbb{N}$ or both by $\mathbb{Z}_2$), the twist map

$\tau\colon V\otimes W\to W\otimes V$ is the same as that for the second index gradings of V, W. If W is graded and $W=F^0W\supseteq F^1W\supseteq\cdots$ is a descending filtration of W, then the associated bigraded vector space is

$$\mathrm{Gr}\, W=\bigoplus_{i,j}(\mathrm{Gr}\, W)_{ij}$$

where $(\mathrm{Gr}\, W)_{ij}=(F^iW/F^{i+1}W)_j$.

By a bigraded Γ-algebra (A, I_0) we mean a bigraded algebra A such that (A, I_0) is a graded Γ-algebra with respect to the second index grading on A, I_0 is homogeneous with respect to the bigrading, and $\gamma_m x\in A_{im,jm}$ for all i, j, m and $x\in I_{ij}$. If a is a graded Γ-algebra and $FA=\{F^iA\}_{i\geqslant 0}$ is a descending filtration of A (and hence $\gamma_m x\in F^{mi}A$ for all i, m and $x\in I\cap F^iA$) then $\mathrm{Gr}\, A$ is a bigraded Γ-algebra, with

$$\gamma_m(x+F^{i+1}A)=\gamma_m x+F^{mi+1}A \qquad \text{for} \quad x\in I\cap F^iA;$$

this is well defined since if $y\in I\cap F^{i+1}A$ then

$$\gamma_m(x+y)=\gamma_m(x)+\sum_{j<m}\gamma_j(x)\,\gamma_{m-j}(y)\in\gamma_m(x)$$
$$+\sum_{j<m}(F^{ji}A)(F^{(m-j)(i+1)}A)\subseteq\gamma_m(x)+F^{mi+1}A.$$

In particular, consider $\mathrm{Tc}V$ with the filtration $F^n\mathrm{Tc}V=\{a\in\mathrm{Tc}V\mid a^0+\cdots+a^{n-1}=0\}$. This is a Hopf Γ-algebra filtration, compatibility with comultiplication (resp. multiplication, antipode, γ_n) following from Lemma 2.3 (resp. Theorem 2.5, Proposition 2.6, the proof of Theorem 3.1). We may identify $F^n\mathrm{Tc}V/F^{n+1}\mathrm{Tc}V$ with T^nV and thus identify $\mathrm{Gr}\,\mathrm{Tc}V$ with $\mathrm{Sh}V$ (previously identified with TV as a vector space) as bigraded Hopf Γ-algebras.

We construct a filtration on an augmented Γ-algebra A which replaces the augmentation filtration (and coincides with it at characteristic 0). For $n\geqslant 0$ we let γ^nA be the span of

$$\left\{\gamma_{i_1}(x_1)\cdots\gamma_{i_m}(x_m)\,\middle|\, m\geqslant 1, \sum i_j\geqslant n, \text{ and } x_j\in I(A) \text{ or } x_j\in A^+ \text{ and } i_j=1\right\}$$

(where $\gamma_1 x$ denotes x if $x\notin I$). Thus in particular $\gamma^0A=A$, $\gamma^1A=A^+$, and $\gamma^2(A)$ is as defined above. We omit the straightforward proof of the following result.

LEMMA 3.2. *If A is an augmented Γ-algebra (respectively Γ-bialgebra, Hopf Γ-algebra) then $\gamma A=\{\gamma^nA\}_{n\geqslant 0}$ is a (descending) filtration of A as a Γ-algebra (resp. Γ-bialgebra, Hopf Γ-algebra).*

We let $Q_\Gamma A$ denote the vector space A^+/γ^2A. This is denoted by QA in [An2], but we reserve the notation QA for $A^+/(A^+)^2$ as in [MM]; of course at characteristic 0 we have $Q_\Gamma A=QA$.

COROLLARY 3.3. *Suppose H is a filtered bialgebra (respectively Hopf algebra, Γ-bialgebra, Hopf Γ-algebra) with descending filtration $JH = \{J^nH\}_{n \geqslant 0}$ where $J^0H = H$, $J^1H = H^+$. If $\varphi: H \to V$ is a linear map such that $\varphi(k + J^2H) = 0$ then, for all n, $\bar{\varphi}(J^nH) \subseteq F^n\mathrm{Tc}V$; also $\bar{\varphi}$ is a bialgebra (resp. Hopf algebra, Γ-bialgebra, Hopf Γ-algebra) map.*

Proof. For the first statement, it suffices to show that $(\bigotimes^m \varphi)\, \Delta_{m-1}(J^nH) = 0$ for $m < n$. But

$$(\bigotimes^m \varphi)\, \Delta_{m-1}(J^nH) \subseteq \sum_{i_1 + \cdots + i_m = n} \varphi(J^{i_1}H) \otimes \cdots \otimes \varphi(J^{i_m}H).$$

Since $m < n$, if $i_1 + \cdots + i_m = n$ then some $i_j \geqslant 2$ and $\varphi(J^{i_j}H) \subseteq \varphi(J^2H) = 0$. The final statement holds by Proposition 2.1 and Theorem 3.1 since $J^1H = H^+$ implies that J^2H contains H^{+2} and (in the Γ-algebra case) γ^2H. Q.E.D.

4. LIE COALGEBRAS AND THEIR COENVELOPING COALGEBRAS

Throughout this section and the next, we assume that either the characteristic $\chi \neq 2$ or if $\chi = 2$ then all degrees are even; the excluded case will be treated in an appendix.

If C is a coalgebra, we get a coalgebra (in the sense of having a comultiplication but not requiring coassociativity or a counit), which we denote by $C^{[-]}$, by taking C to be the underlying vector space and defining comultiplication by

$$[\Delta] = (1 - \tau)\, \Delta: C \to C \otimes C.$$

We define a *Lie coalgebra* to be a coalgebra (K, δ) (not necessarily coassociative or counital) for which there is a coalgebra C (coassociative and counital) and a surjective coalgebra map $\omega: C^{[-]} \to K$. Thus a Lie coalgebra is a (graded) vector space K and linear map $\delta: K \to K \otimes K$ for which there is a coalgebra (C, Δ, ε) and surjective linear map $\omega: C \to K$ such that $\delta\omega = (\omega \otimes \omega)(1 - \tau)\, \Delta$. This, the dual analogue of the definition of a Lie algebra by embedding in A^-, as given in [MM], is the definition of Lie coalgebra given by André [An2] and also used by Nichols [Ni]. As in [An2] we call any pair (C, ω) with the above properties a *cover* of K. There is another definition of Lie coalgebra, given by (co)identities, which is a dual form of the definition of Lie algebras by identities; this definition was given in [An] and [Mi]. It has been proved by Michaelis [Mi] that, unlike the Lie algebra case, the definition by identities is actually more general than that by covers. (However, the classes of coenveloping coalgebras UcK are the same in the two cases.) He has also proved (as a consequence of Ado's theorem for Lie algebras and the local finiteness (Fundamental Theorem) of coalgebras) that (K, δ) is a Lie coalgebra, in the sense of having a cover, if and only if it is a Lie coalgebra in the sense of satisfying the identities and is locally finite (that is, its finitely generated Lie subcoalgebras are finite dimensional). It may be noted that, conversely, Ado's theorem is an easy consequence of (the "if" part of) Michaelis' theorem and the local finiteness of coalgebras. It follows from Michaelis' theorem that subcoalgebras of Lie coalgebras (in our sense) are themselves Lie coalgebras (in our sense).

A cover (C, ω) of a Lie coalgebra (K, δ) is called a *universal cover*, or *coenveloping coalgebra*, of K if for any coalgebra D and Lie coalgebra map $\varphi: D^{[-]} \to K$ there is a unique coalgebra map $\theta: D \to C$ such that $\omega\theta = \varphi$. Coenveloping coalgebras satisfy the obvious uniqueness property.

If K is a Lie coalgebra we let TcK denote the cofree coalgebra on the underlying vector space of K, and similarly for ShK and, later, BK. We say that an element f in TcK has the *Lie property* if

$$(1-\tau)(\pi \otimes \pi)\,\Delta f = \delta\pi f, \qquad \text{that is,} \quad (1-\tau)\, f^2 = \delta f^1.$$

Let UcK be the (unique) largest subcoalgebra of TcK contained in the set of all elements with the Lie property.

PROPOSITION 4.1. *If D is a coalgebra and $\varphi: D^{[-]} \to K$ is a Lie coalgebra map then $\bar{\varphi}D \subseteq$ UcK. Moreover* UcK *together with the restriction of π to* UcK *(also denoted π) is a coenveloping coalgebra of K.*

Proof. That $\pi: (\mathrm{Uc}K)^- \to K$ is a Lie coalgebra map follows from the fact that the elements of UcK have the Lie property. We have $\bar{\varphi}D \subseteq \mathrm{Uc}K$ since, for $d \in D$,

$$\begin{aligned}\delta\pi\bar{\varphi}d = \delta\varphi d &= (\varphi \otimes \varphi)(1-\tau)\,\Delta d = (1-\tau)(\varphi \otimes \varphi)\,\Delta d \\ &= (1-\tau)(\pi \otimes \pi)(\bar{\varphi} \otimes \bar{\varphi})\,\Delta d = (1-\tau)(\pi \otimes \pi)\,\Delta\bar{\varphi}d.\end{aligned}$$

Hence $\bar{\varphi}$, with codomain restricted to UcK, is the unique coalgebra map $\theta: D \to \mathrm{Uc}K$ such that $\pi\theta = \varphi$. Finally, $\pi\mathrm{Uc}K = K$ since by definition K has a cover (C, ω) with ω surjective, and $\pi\bar{\omega} = \omega$. Q.E.D.

PROPOSITION 4.2. *If K is a Lie coalgebra then* UcK *is a sub Hopf Γ-algebra of* TcK *(with $I_0(\mathrm{Uc}K) = (\mathrm{Uc}K)_0^+$).*

Proof. If $f, g \in \mathrm{Tc}K$ have the Lie property then so does fg. Indeed we have $\delta f^1 = (1-\tau)\, f^2$, $\delta g^1 = (1-\tau)\, g^2$, and, by Theorem 2.5,

$$fg = (\varepsilon f)(\varepsilon g) + ((\varepsilon f)\, g^1 + f^1(\varepsilon g)) + ((\varepsilon f)\, g^2 + f^1 g^1 + f^2(\varepsilon g)) + \cdots$$

and hence

$$\begin{aligned}\delta(fg)^1 &= (\varepsilon f)\,\delta g^1 + (\varepsilon g)\,\delta f^1\\ &= (\varepsilon f)(1-\tau)\,g^2 + (\varepsilon g)(1-\tau)\,f^2 = (1-\tau)(fg)^2\end{aligned}$$

since the shuffle product f^1g^1 is symmetric. If $f \in I(\mathrm{Uc}K)$ and $n \geqslant 0$ then $\gamma_n f$ has the Lie property since $(\gamma_n f)^1 = 0 = (\gamma_n f)^2$ if $n > 2$, and $(\gamma_2 f)^1 = 0$ and $(\gamma_2 f)^2 = \gamma_2 f^1$ is symmetric. If f has the Lie property then so does Sf since, by Proposition 2.6,

$$\delta(Sf)^1 = \delta(-f^1) = (\tau - 1)\,f^2 = (1-\tau)\,\tau f^2 = (1-\tau)(Sf)^2.$$

Therefore the sub S-invariant Γ-algebra of $\mathrm{Tc}K$ generated by $\mathrm{Uc}K$ is a subcoalgebra all of whose elements have the Lie property, hence equals $\mathrm{Uc}K$.
Q.E.D.

Let (K, δ) be a Lie coalgebra. For $a \in \mathrm{Tc}K$, $n > 0$ and j with $0 \leqslant j \leqslant n-1$, we consider the following condition (we write 1^m in place of $\otimes^m 1$):

$$(1^j \otimes \delta \otimes 1^{n-1-j})\,a^n = (1^j \otimes (1-\tau) \otimes 1^{n-1-j})\,a^{n+1}. \tag{4.1}$$

We call $a \in \mathrm{Tc}K$ *Lie-symmetric* if a satisfies (4.1) for all n, j. Notice that the case $n = 1$, $j = 0$ is the Lie property of a.

LEMMA 4.3. *$\mathrm{Uc}K$ equals the set D of Lie-symmetric elements of $\mathrm{Tc}K$.*

Proof. To show that $D \subseteq \mathrm{Uc}K$ it suffices to show that D is a subcoalgebra. Let $\{v_j \mid j \in J\}$ be a basis of K, and consider the associated operators R_j, L_j $(j \in J)$ defined in (1.2). If $a \in D$, $i \in J$, $n \geqslant 1$, and $0 \leqslant j \leqslant n-1$ we have

$$\begin{aligned}(1^j \otimes \delta \otimes 1^{n-1-j})(R_i a)^n &= (1^j \otimes \delta \otimes 1^{n-1-j})(1^n \otimes \lambda_i)\,a^{n+1}\\ &= (1^{n+1} \otimes \lambda_i)(1^j \otimes \delta \otimes 1^{n-j})\,a^{n+1}\\ &= (1^{n+1} \otimes \lambda_i)(1^j \otimes (1-\tau) \otimes 1^{n-j})\,a^{n+2}\\ &= (1^j \otimes (1-\tau) \otimes 1^{n-1-j})(1^{n+1} \otimes \lambda_i)\,a^{n+2}\\ &= (1^j \otimes (1-\tau) \otimes 1^{n-1-j})(R_i a)^{n+1}.\end{aligned} \tag{4.2}$$

Thus $R_i a \in D$, and similarly $L_i a \in D$. Therefore, by Lemma 1.2, D is a subcoalgebra, and $D \subseteq \mathrm{Uc}K$. To show that $\mathrm{Uc}K \subseteq D$, suppose for some $n \geqslant 1$ that (4.1) holds for every $a \in \mathrm{Uc}K$ and for all j, $0 \leqslant j \leqslant n-1$ (this is true for $n = 1$ by the Lie property). Suppose $a \in \mathrm{Uc}K$ and $0 \leqslant j \leqslant n$. To show that (4.1) holds for $n+1$ in place of n, it suffices, when $j \leqslant n-1$, to show, for each $i \in J$, the equality of the result of applying $(1^{n+1} \otimes \lambda_i)$ to the two sides.

But the equality is given by (4.2) since $R_i a$, being in UcK, satisfies (4.1) for n. In case $j=n$, a similar argument, using L_i instead of R_i, gives the result. Q.E.D.

We give some definitions and notation, in which V continues to be a (graded) vector space. We say that $f \in \mathrm{Tc}V$ is *symmetric* if each f^n is symmetric (of course in the graded sense if V is graded). We make the vector space V into a Lie coalgebra, which we denote by $[V]$, by taking $\delta=0$; this has a cover $(k+V, \omega)$ with $1 \in k$ grouplike, the elements of V primitive, and ω the projection on V. By Lemma 4.3, V (identified as before with a subspace of TcV) is contained in Uc$[V]$. We call a Lie coalgebra with $\delta=0$ *abelian.*

We write CcV for the cofree cocommutative coalgebra on V, regarded as the sum of all cocommutative subcoalgebras of TcV; the canonical map to V is the restriction of π, which we also denote by π.

PROPOSITION 4.4. $\mathrm{Cc}V=\mathrm{Uc}[V]=\{f \in \mathrm{Tc}V \mid f \text{ symmetric}\}$. *In particular,* Uc$[V]$ *is cocommutative, and* CcV *is a sub Hopf Γ-algebra of* TcV.

Proof. $\mathrm{Uc}[V]=\{f \in \mathrm{Tc}V \mid f \text{ symmetric}\}$ by Lemma 4.3. That $\mathrm{Cc}V= \{f \in \mathrm{Tc}V \mid f \text{ symmetric}\}$ was proved in [BL] for V ungraded and the proof there remains valid in the graded case when the sign convention is taken into account. Q.E.D.

Let $\{x_i\}_{i \in I} \cup \{y_j\}_{j \in J}$ be a basis of the vector space V, where I and J are ordered, each x_i is even homogeneous, and each y_j is odd homogeneous. Recall that $\gamma_s x_i = \bigotimes^s x_i$, and $\cdot$ denotes shuffle product.

LEMMA 4.5. *Let the basis of V be as above. Then*

$$\{\gamma_{e_1} x_{i_1} \cdot \cdots \cdot \gamma_{e_s} x_{i_s} \cdot y_{j_1} \cdot \cdots \cdot y_{j_t} \mid i_1 < \cdots < i_s; j_1 < \cdots < j_t; \\ e_m \geqslant 1 \ (m=1,..., s); s, t \geqslant 0\} \quad (4.3)$$

is a basis of the space of symmetric elements of TV.

Proof. It is easy to see that these elements are symmetric and linearly independent. To see that they span, note that in a symmetric tensor all the (noncommutative) tensor monomials with the same multiplicities for the x_i's and y_j's have the same coefficient up to an appropriate sign, no y_j can have multiplicity >1, and the sum of these monomials (each times the appropriate sign) is one of the desired shuffle products. Q.E.D.

The intersection $(\mathrm{Tc}V)^1 \cap \mathrm{Cc}V = \mathrm{Sh}V \cap \mathrm{Cc}V$ is an irreducible cocommutative Hopf Γ-algebra. This is obviously the unique maximal cocommutative subcoalgebra of ShV. In the ungraded case, Sweedler [Sw] denotes this BV; we use the same notation in the graded case.

COROLLARY 4.6. *The cofree pointed irreducible cocommutative coalgebra BV on V is (realized as) the sub Hopf Γ-algebra of* ShV *consisting of all the symmetric elements of* ShV. *The set displayed in* (4.3) (*obtained from a basis of homogeneous elements of V*) *is a basis of BV.*

The free Γ-algebra on V (see [Ro, Sc, GL, BO]) is often denoted ΓV. This is isomorphic to the Γ-algebra of divided power polynomials in a basis (of homogeneous elements) of V [GL] (we allow generators of degree 0, and we allow gradings by $\mathbb{Z}_2$). Since the monomials correspond to the elements displayed in (4.3), we have the following result.

COROLLARY 4.7. *$BV \cong \Gamma V$ as Γ-algebras; more precisely, the canonical map of ΓV to BV determined by the inclusion map of V into BV is a Γ-algebra isomorphism.*

We identify BV with ΓV via this map. Writing

$$V(\text{odd}) = \sum_{i \text{ odd}} V_i, \qquad V(\text{even}) = \sum_{i \text{ even}} V_i,$$

we also see that $BV \cong B(V(\text{even})) \otimes B(V(\text{odd}))$ as Hopf Γ-algebras, and that $B(V(\text{odd})) \cong E(V(\text{odd}))$, the exterior algebra on $V(\text{odd})$.

COROLLARY 4.8. *Suppose k is algebraically closed, and let G denote the group of grouplikes of* TcV. *Then the linear mapping*

$$BV \otimes kG \to \mathrm{Cc}V, \qquad b \otimes g \mapsto bg \qquad (b \in BV,\ g \in G)$$

is an isomorphism of Hopf algebras; in particular, CcV *has a basis*

$$\{z \cdot e^v = z + z \cdot v + z \cdot (v \otimes v) + \cdots \mid z \in Z,\ v \in V_0\}$$

where Z is the basis of BV of (4.3).

Proof. Write $W = \sum_{i \neq 0} V_i$. Since right adjoint functors preserve products, the canonical map $\mathrm{Cc}V_0 \otimes \mathrm{Cc}W \to \mathrm{Cc}(V_0 \oplus W)$ induced by $1 \otimes \varepsilon$ and $\varepsilon \otimes 1$ is an isomorphism (of Hopf algebras; it can be seen by composing with projections that this product map equals the bar of $\pi(1 \otimes \varepsilon) \oplus \pi(\varepsilon \otimes 1)$). It follows from the definition of multiplication in TcV that this map sends tensor products to products. Since $W_0 = 0$, $(\mathrm{Tc}W)_m \subseteq (\mathrm{Sh}W)_0 + \cdots + (\mathrm{Sh}W)_m$ for all m, $\mathrm{Tc}W \subseteq \mathrm{Sh}W$, and $\mathrm{Cc}W = BW$. (If V is $\mathbb{Z}_2$-graded then $W = V_1$ and again $\mathrm{Cc}W = BW$, by reduction to the case W finite dimensional and use of Lemma 4.5.) By Theorem 8.1.5 of [Sw], $(\mathrm{Cc}V_0)^1 \otimes kG(\mathrm{Cc}V_0) \cong \mathrm{Cc}V_0$ by a Hopf algebra isomorphism sending tensor products to products. But $(\mathrm{Cc}V_0)^1 = BV_0$ and $G(\mathrm{Cc}V_0) = G = \{e^v \mid v \in V_0\}$ by Lemma 2.2. Q.E.D.

If k is not algebraically closed, the above holds for the cofree pointed cocommutative coalgebra in place of CcV.

Suppose K is a Lie coalgebra. We define a descending filtration $L\mathrm{Uc}K = \{L^n\mathrm{Uc}K\}_{n \geqslant 0}$, called the *Lie filtration*, on $\mathrm{Uc}K$ by taking, for $n \geqslant 0$,

$$L^n\mathrm{Uc}K = (F^n\mathrm{Tc}K) \cap \mathrm{Uc}K \qquad (= \{a \in \mathrm{Uc}K \mid a^i = 0, i < n\}).$$

Since $F\mathrm{Tc}K$ is a filtration of $\mathrm{Tc}K$ as a Hopf Γ-algebra, it follows that $L\mathrm{Uc}K$ is a filtration of $\mathrm{Uc}K$ as a Hopf Γ-algebra. We remark that (with Λ the wedge function of [Sw]) $F^n\mathrm{Tc}V = \Lambda^{n-1}(F^2\mathrm{Tc}V)$ for $n \geqslant 2$ (and $F^2\mathrm{Tc}V = (\mathrm{Ker}\,\varepsilon) \cap \mathrm{Ker}\,\pi)$) and hence $L^n\mathrm{Uc}K = \Lambda^{n-1}(L^2\mathrm{Uc}K)$ for $n \geqslant 2$ (and $L^2\mathrm{Uc}K = (\mathrm{Ker}\,\varepsilon) \cap (\mathrm{Ker}\,\pi)$ for the ε, π of $\mathrm{Uc}K$). We denote the (bi)graded Hopf Γ-algebra corresponding to $L\mathrm{Uc}K$ by $\mathrm{Gr}_L\mathrm{Uc}K$. We recall that we have identified $\mathrm{Gr}\,\mathrm{Tc}V$ with $\mathrm{Sh}\,V$.

We now give a dual version of the Poincaré–Birkhoff–Witt theorem. Suppose K is a Lie coalgebra and let ι denote the inclusion map of $\mathrm{Uc}K$ to $\mathrm{Tc}K$; we have $\iota L^n\mathrm{Uc}K \subseteq F^n\mathrm{Tc}K$ and hence $\mathrm{Gr}\,\iota\colon \mathrm{Gr}_L\mathrm{Uc}K \to \mathrm{Gr}\,\mathrm{Tc}K$ is defined.

THEOREM 4.9. $\mathrm{Gr}\,\iota\colon \mathrm{Gr}_L\mathrm{Uc}K \to \mathrm{Gr}\,\mathrm{Tc}K = \mathrm{Sh}K$ *has its image equal to* BK, *and with its codomain restricted to* BK, $\mathrm{Gr}\,\iota\colon \mathrm{Gr}_L\mathrm{Uc}K \cong BK$, *an isomorphism of (bi)graded Hopf* Γ*-algebras.*

Proof. If $a \in L^n\mathrm{Uc}K$ then by the Lie-symmetry of a (Lemma 4.3), a^n is symmetric. Hence $\mathrm{Gr}\,\iota(\mathrm{Gr}_L\mathrm{Uc}K) \subseteq BK$. It is obvious that $\mathrm{Gr}\,\iota$ is injective and a bigraded Hopf Γ-algebra map. We have $K \subseteq \mathrm{Gr}\,\iota(\mathrm{Gr}_L\mathrm{Uc}K)$ because $\pi(\mathrm{Uc}K) = K$ and $1 \in \mathrm{Uc}K$. But K generates BK as a Γ-algebra and hence $BK \subseteq \mathrm{Gr}\,\iota(\mathrm{Gr}_L\mathrm{Uc}K)$. Q.E.D.

We now discuss briefly the relationship between Lie algebras and Lie coalgebras. By an envelope of a Lie algebra we mean a pair (A, θ) where A is an algebra and θ is an injective Lie algebra map of L into the Lie algebra A^- associated with A. If (K, δ) is a Lie coalgebra, with a cover (C, ω), then it is easy to see, by dualizing the defining diagram of a cover, that the dual space K^*, with multiplication the restriction to $K^* \otimes K^*$ of δ^*, is a Lie algebra (obviously independent of the choice of cover) and that (C^*, ω^*) is an envelope of K^*. Starting with a Lie algebra L we now construct a Lie coalgebra L^0. (The reader is warned that, just as our notion of Lie coalgebra is more restrictive than that of Michaelis [Mi], our definition of L^0 differs from that given (for the ungraded case) by him. We shall see below that our L^0 equals the locally finite part $\mathrm{Loc}\,L^0_{\mathrm{M}}$ of L^0_{M}, where we denote Michaelis' construction by L^0_{M}.)

Let $\eta\colon L \to UL$ be the canonical injection. We define

$$L^0 = \eta^*((UL)^0),$$

and denote by η^0 the map η^* with its domain restricted to $(UL)^0$ and its codomain to L^0. Thus if L is regarded as a subspace of UL, L^0 consists of the elements of L^* which can be extended to representative functions on UL.

PROPOSITION 4.10. *If L is a Lie algebra then there exists a unique Lie coalgebra structure (L^0, δ) on L^0 such that $((UL)^0, \eta^0)$ is a cover. For $x, y \in L$ and $a \in L^0$,*

$$\delta a(x \otimes y) = a[x, y].$$

If A is an algebra and $\theta\colon L \to A^-$ is a Lie algebra map then $\theta^*(A^0) \subseteq L^0$, and $\theta^*|_{A^0}$, with codomain restricted to L^0, is a Lie coalgebra map.

Proof. To prove the first assertion it suffices to show (for the Δ in $(UL)^0$) that $(\eta^0 \otimes \eta^0)(1-\tau)\,\Delta(\ker \eta^0) = 0$. If $f \in (UL)^0$ and $x, y \in L$ we have

$$\begin{aligned}
&((\eta^0 \otimes \eta^0)(1-\tau)\,\Delta f)(x \otimes y)\\
&= \Big(\sum_{(f)} (1-\tau)(\eta^0 f_{(1)} \otimes \eta^0 f_{(2)})\Big)(x \otimes y)\\
&= \sum_{(f)} \{f_{(1)}(\eta x)\, f_{(2)}(\eta y) - f_{(2)}(\eta x)\, f_{(1)}(\eta y)\}\\
&= f(\eta x \eta y - \eta y \eta x) = f(\eta[x, y]),
\end{aligned}$$

which equals 0 if $\eta^0 f = 0$. This also proves the second assertion, since $\delta(\eta^0 f) = (\eta^0 \otimes \eta^0)(1-\tau)\,\Delta f$. Finally, if A, θ satisfy the hypotheses and $\bar{\theta}\colon UL \to A$ is the corresponding algebra map, then $f \in A^0$ implies that $\theta^* f = \eta^* \bar{\theta}^* f \in L^0$. The final assertion holds since $\theta^*|_{A^0} = \eta^0 \bar{\theta}^0$. Q.E.D.

COROLLARY 4.11. *If L is finite dimensional then $L^0 = L^*$.*

Proof. Since $A^0 = A^*$ if A is finite dimensional, this follows from the last sentence of the proposition and the Ado–Iwasawa theorem (that theorem also holds in the graded case—see [Rs]; the characteristic 2 case can be included in [Rs] with the use of the PBW theorem at characteristic 2 [MM, Sj]).

It also follows from the last sentence of the proposition that if $\sigma\colon L \to M$ is a Lie algebra map then $\sigma^* M^0 \subseteq L^0$, and hence we have a functor $(\ )^0$: Lie Alg $\to$ (Lie Coalg)$^{\mathrm{op}}$. It can be show that this is left adjoint to the functor $(\ )^*$: Lie Coalg$^{\mathrm{op}}$ $\to$ Lie Alg.

We now give an intrinsic characterization of the elements of L^0. We remark that Michaelis [Mi] has shown that $\mathrm{Loc}(L^0_{\mathrm{M}}) = \{f \in L^* \mid \mathrm{Ker}\, f$ contains a cofinite ideal of $L\}$.

PROPOSITION 4.12. *$L^0 = \{f \in L^* \mid \operatorname{Ker} f$ contains a cofinite ideal of $L\}$.*

Proof. Suppose $f = \eta^0 h$ for some $h \in (UL)^0$. Then there exists a cofinite ideal J of UL with $hJ = 0$. Therefore $\{x \in L \mid \eta x \in J\}$ is a cofinite ideal of L annihilated by f. Conversely, suppose $f \in L^*$ and J is a cofinite ideal of L such that $fJ = 0$. Then f induces $g \in (L/J)^*$. By the Ado–Iwasawa theorem, L/J has a faithful finite-dimensional module M; let ρ denote the corresponding representation of $U(L/J)$. We may regard L/J as a subspace of $\operatorname{End} M$. Then g extends to a linear functional h on $\operatorname{End} M$, and $h\rho \in U(L/J)^0$. Let θ denote the composite map $L \to L/J \to U(L/J)$. Then $f = \theta^*(h\rho)$, and so $f \in L^0$. Q.E.D.

Michaelis has shown (in the ungraded case) by using adjoint functors (part of the proof is given in [Mi]) that if L is a Lie algebra then $(UL)^0 \cong \mathrm{Uc}(L^0_{\mathrm{M}})$ as Hopf algebras. He has also shown that $\mathrm{Uc}(L^0_{\mathrm{M}}) = \mathrm{Uc}(\operatorname{Loc} L^0_{\mathrm{M}})$; thus $(UL)^0 \cong \mathrm{Uc}(L^0)$. We now give a different proof of the latter fact, together with a useful formula.

PROPOSITION 4.13. *Let L be a Lie algebra and let η denote the canonical injection $L \to UL$. Then $\overline{\eta^0}\colon (UL)^0 \to \mathrm{Uc}(L^0)$ is an isomorphism of Hopf algebras. Moreover*

$$(\overline{\eta^0} f)^n (x_1 \otimes \cdots \otimes x_n) = f((\eta x_1) \cdots (\eta x_n))$$

for all $f \in (UL)^0$, $n \geqslant 0$, and $x_1, \ldots, x_n \in L$.

Proof. We have $\eta^0 k = 0$ since $\eta^0 \varepsilon_{UL} = \varepsilon_{UL} \eta = 0$. Also $\eta^0(((UL)^0)^+)^2 = 0$ since if $d, f \in (UL)^0$ and $d(1) = f(1) = 0$ then $\eta^0(d * f)\, x = (d * f)(\eta x) = d(1) f(x) + d(x) f(1) = 0$ for all $x \in L$. Hence $\overline{\eta^0}$ is a Hopf algebra map by Proposition 2.1. For $f \in (UL)^0$ and $x_1, \ldots, x_n \in L$ we have

$$\begin{aligned}(\overline{\eta^0} f)^n (x_1 \otimes \cdots \otimes x_n) &= \sum_{(f)} (\eta^0 f_{(1)} \otimes \cdots \otimes \eta^0 f_{(n)})(x_1 \otimes \cdots \otimes x_n) \\ &= \sum_{(f)} (f_{(1)} \eta x_1) \cdots (f_{(n)} \eta x_n) = f((\eta x_1) \cdots (\eta x_n))\end{aligned}$$

since f is a representative function. It follows from this result that $\overline{\eta^0}$ is injective. For $a \in \mathrm{Uc}(L^0)$ we define $a' \in (UL)^*$ by

$$a'((\eta x_1) \cdots (\eta x_n)) = a^n (x_1 \otimes \cdots \otimes x_n) \qquad (x_1, \ldots, x_n \in L).$$

This is well defined, since if $x_1, \ldots, x_m$, $x, y \in L$ and z (resp. w) denotes $x_1 \otimes \cdots \otimes x_j$ (resp. $x_{j+1} \otimes \cdots \otimes x_m$) then

$$\begin{aligned}a^{m+2}&(z\otimes(x\otimes y-y\otimes x)\otimes w)-a^{m+1}(z\otimes[xy]\otimes w)\\&=\{(1^j\otimes(1-\tau)\otimes 1^{m-j})\,a^{m+2}\\&\quad-(1^j\otimes\delta\otimes 1^{m-j})\,a^{m+1}\}(z\otimes x\otimes y\otimes w)=0.\end{aligned}$$

Also $a'\in(UL)^0$ since if $x_i\in L$ and $y_i=\eta x_i$ for $i=1,\ldots,m+n$ then

$$\begin{aligned}a'(y_1\cdots y_{m+n})&=a^{m+n}(x_1\otimes\cdots\otimes x_{m+n})\\&=\sum_{(a)}(a_{(1)}{}^m\otimes a_{(2)}{}^n)(x_1\otimes\cdots\otimes x_{m+n})\\&=\sum_{(a)}(a_{(1)})'(y_1\cdots y_m)(a_{(2)})'(y_{m+1}\cdots y_{m+n}).\end{aligned}$$

It follows that $a\in\overline{\eta^0}(UL)^0$, namely, $a=\overline{\eta^0}a'$, and so $\overline{\eta^0}$ is surjective. Q.E.D.

LEMMA 4.14. *Suppose K is a Lie coalgebra. Then the evaluation map $K\to K^{**}$ has its image in K^{*0}, and with its codomain restricted to K^{*0}, is a Lie coalgebra map.*

Proof. Suppose (C,ω) is a cover of K, and let λ,ψ denote the evaluation maps $\lambda\colon K\to K^{**}$ and $\psi\colon C\to C^{*0}$. Then for $c\in C$, $f\in K^*$ we have $(\lambda\omega c)\,f=f(\omega c)=(\omega^*f)\,c=(\psi c)\,\omega^*f=(\omega^{*0}\psi c)\,f$ and hence $\lambda\omega=\omega^{*0}\psi$ and $\lambda K\subseteq K^{*0}$. Now regard λ as having its codomain restricted to K^{*0}. Then λ is a Lie coalgebra map, since for $a\in K$ and $f,h\in K^*$ we have $(\delta^{*0}\lambda a)(f\otimes h)=(\lambda a)[f,h]=(\lambda a)\,\delta^*(f\otimes h)=(\lambda a)((f\otimes h)\,\delta)=(f\otimes h)\,\delta a=((\lambda\otimes\lambda)\,\delta a)(f\otimes h)$ (here δ^{*0} denotes the Lie comultiplication on K^{*0} arising by Proposition 4.10 from the Lie algebra (K^*,δ^*)). Q.E.D.

5. THE EMBEDDING THEOREM FOR Γ-BIALGEBRAS

If A is a Γ-bialgebra, let $\zeta=\zeta_A$ denote the map of A to $Q_\Gamma A=A^+/\gamma^2A$ defined by $\zeta a=(1-\varepsilon)\,a+\gamma^2A$ $(a\in A)$.

LEMMA 5.1. *$Q_\Gamma A$ has a unique Lie coalgebra structure such that (A,ζ) is a cover.*

Proof. We must have, for all $a\in A$, $\delta(\zeta a)=(\zeta\otimes\zeta)(1-\tau)\,\Delta a$; it remains to prove that this gives a well-defined mapping δ, that is, $\operatorname{Ker}\zeta\subseteq\operatorname{Ker}(\zeta\otimes\zeta)(1-\tau)\,\Delta$. We have $\operatorname{Ker}\zeta=k+\gamma^2A$. First $k\subseteq\operatorname{Ker}(1-\tau)\,\Delta$. Next, if $x\in A^+$ then $\Delta x\equiv 1\otimes x+x\otimes 1 \bmod A^+\otimes A^+$ and so if $x,y\in A^+$ then

$$\Delta(xy)=\Delta x\,\Delta y\equiv 1\otimes xy+xy\otimes 1+x\otimes y+y\otimes x \mod(A^+\otimes\gamma^2A+\gamma^2A\otimes A^+).$$

Therefore $(\zeta\otimes\zeta)(1-\tau)\,\Delta(xy)=0$. Finally, suppose $x\in I(A)$ (and hence even) and $i\geqslant 2$. We have $\Delta\gamma_i x=\gamma_i\,\Delta x$, $\Delta x=1\otimes x+x\otimes 1+a_3\otimes b_3+\cdots+a_s\otimes b_s$ where a_j, b_j are homogeneous. Therefore

$$\gamma_i\,\Delta x=\sum(\gamma_{j_2}x)\,a_3^{j_3}\cdots a_s^{j_s}\otimes(\gamma_{j_1}x)(\gamma_{j_3}b_3)\cdots(\gamma_{j_s}b_s)$$

where the sum is over all $(j_1,\dots,j_s)$ with $j_1+\cdots+j_s=i$ and with $j_t<2$ if b_t (and hence also a_t) is odd (in which case $\gamma_1 b_t$ denotes b_t and $\gamma_0 b_t$ denotes 1). Each summand is in $\operatorname{Ker}(\zeta\otimes\zeta)$ except possibly when $i=2$ and $(j_1,\dots,j_s)=(1,1,0,\dots,0)$, in which case the summand equals $x\otimes x\in\operatorname{Ker}(1-\tau)$. Q.E.D.

LEMMA 5.2. *Suppose A is a bigraded Γ-bialgebra such that $A_{00}=k$, $A_{0j}=0$ for $j\neq 0$ and $I_0(A)=A_0^+$ $(=\sum_{i>0}A_{i0})$. Then the restriction of ζ to the space PA of primitives of A is injective.*

At characteristic 0 this result is essentially contained in the bigraded version of Proposition 4.17 of [MM]. That proof was extended to the divided power case by André [An2, pp. 33–34]. André treats the graded case, but the proof extends to the bigraded case stated here (with the second index grading by $\mathbb{N}$ or $\mathbb{Z}_2$) without difficulty. We make two further remarks on the proof in [An2]. The first part of the proof shows that A is a union of finitely generated Hopf Γ-algebras; but this follows immediately from the finiteness theorem for coalgebras [Sw, p. 46]. Also, on page 34, line 1 of [An2] André appears to need a hypothesis on $\gamma_m(\gamma_n x)$, such as (3.1.6) (which he omitted from his list of axioms for a Γ-algebra), since he uses the fact that if A is generated as a Γ-algebra by a single element x of even degree then A is spanned by $\{\gamma_n x\mid n=0,1,2,\dots\}$.

If A is a Γ-bialgebra with $I_0=A_0^+$ then the graded object associated to the descending filtration $\gamma A=\{\gamma^n A\}_{n\geqslant 0}$, which we denote $\operatorname{Gr}_\gamma A$, satisfies the hypothesis of Lemma 5.2. For $\zeta=\zeta_A$ as above, it follows from Corollary 3.3 that $\bar\zeta\gamma^n A\subseteq L^n\mathrm{Uc}Q_\Gamma A$, that is, $\bar\zeta$ gives a map of filtered Γ-bialgebras. Hence we may consider the bigraded Hopf Γ-algebra map $\operatorname{Gr}\bar\zeta\colon\operatorname{Gr}_\gamma A\to\operatorname{Gr}_L UQ_\Gamma A$ associated to $\bar\zeta$. We also write (as for any descending filtration) $\gamma^\infty A=\bigcap_{n>0}\gamma^n A$.

THEOREM 5.3. *Suppose A is a Γ-bialgebra with $I_0(A)=A_0^+$, and ζ is the canonical map to the associated Lie coalgebra $K=Q_\Gamma A$. Then $\bar\zeta\colon A\to\mathrm{Uc}K$ is a Γ-bialgebra map,* $\operatorname{Ker}\bar\zeta=\gamma^\infty A$, $\operatorname{Im}\bar\zeta$ *is dense (in the subspace topology induced from $\overline{TV}$),*

$$\operatorname{Gr}\bar\zeta\colon\operatorname{Gr}_\gamma A\to\operatorname{Gr}_L\mathrm{Uc}K$$

is an isomorphism of Hopf Γ-algebras, and $\operatorname{Gr}_\gamma A\cong BK$ *as Hopf Γ-algebras (via the isomorphism* $\operatorname{Gr}\iota\operatorname{Gr}\bar\zeta$, *with ι as in the dual PBW theorem* (4.9)).

Proof. We write $\mathrm{Gr}_i A = \gamma^i A/\gamma^{i+1} A$ $(= \bigoplus_j (\mathrm{Gr}_\gamma A)_{ij})$. It follows from the definition of $\gamma^n A$ that if $i > 1$ then $\mathrm{Gr}_i A \subseteq \gamma^2 \mathrm{Gr}_\gamma A$. Therefore, by Lemma 5.2, if $0 \neq x \in P(\mathrm{Gr}_\gamma A)$ then the component x_1 of x in $\mathrm{Gr}_1 A$ $(=K)$ is nonzero. Pick $y \in \gamma^1 A$ such that $y + \gamma^2 A = x_1$. Then $(\mathrm{Gr}\,\bar\zeta)\, x_1 = 0$ if and only if $\bar\zeta y \in L^2\mathrm{Uc}K$, that is, if and only if $\pi\bar\zeta y = 0$. But $\pi\bar\zeta y = \zeta y = x_1 \neq 0$. Therefore the restriction to $P(\mathrm{Gr}_\gamma A)$ of $\mathrm{Gr}\,\bar\zeta$ is injective, and hence (by Proposition B3.9 of [MM] or Lemma 11.0.1 of [Sw]) $\mathrm{Gr}\,\bar\zeta$ is injective. Since $\bar\zeta$ is filtration-preserving and $L^\infty\mathrm{Uc}K = 0$, we have $\gamma^\infty A \subseteq \mathrm{Ker}\,\bar\zeta$. On the other hand, if $a \in \mathrm{Ker}\,\bar\zeta$ and $a \notin \gamma^\infty A$, then, for some i, $a \in \gamma^i A$ and $a \notin \gamma^{i+1} A$. But then $(\mathrm{Gr}\,\bar\zeta)(a + \gamma^{i+1} A) \neq 0$, and so $\bar\zeta a \notin L^{i+1}\mathrm{Uc}K$, a contradiction. Therefore $\mathrm{Ker}\,\bar\zeta = \gamma^\infty A$. To show that $\mathrm{Gr}\,\bar\zeta$ is surjective, we write $W = L^1\mathrm{Uc}K/L^2\mathrm{Uc}K$ and suppose $\bar f = f + L^2\mathrm{Uc}K \in W$. Then there exists $a \in \gamma^1 A$ such that $\zeta a = \pi f$, and we have $(\mathrm{Gr}\,\bar\zeta)(a + \gamma^2 A) = \bar\zeta a + L^2\mathrm{Uc}K = \bar f$ (since $\pi\bar\zeta a = \zeta a$). Therefore $W \subseteq (\mathrm{Gr}\,\bar\zeta)\,\mathrm{Gr}_\gamma A$. By Theorem 4.9, $\mathrm{Gr}_L \mathrm{Uc}K \cong BK$ under an isomorphism sending W to the canonical copy of K in BK. Therefore W generates $\mathrm{Gr}_L\mathrm{Uc}K$ as a Γ-algebra, and so $\mathrm{Gr}\,\bar\zeta$ is surjective. It is easily seen that this surjectivity is equivalent to the stated density property. Q.E.D.

COROLLARY 5.4 (The Milnor–Moore–André Theorem). *If the Γ-bialgebra A is connected then $\bar\zeta\colon A \to \mathrm{Uc}Q_\Gamma A$ is an isomorphism.*

Proof 1. Write $K = Q_\Gamma A$. Since A is connected, $K_0 = 0$ and $\mathrm{Uc}K$ is also connected. Then the filtrations γA and $L\mathrm{Uc}K$ are complete and hence [MM, p. 240] $\mathrm{Gr}\,\bar\zeta$ being an isomorphism implies that $\bar\zeta$ is an isomorphism. Q.E.D.

Proof 2. Since $\gamma^n A \subseteq \sum_{i \geqslant n} A_i$, $\gamma^\infty A = 0$, and $\bar\zeta$ is injective. Suppose $(\bar\zeta A)_n \neq (\mathrm{Uc}K)_n$. Since (by connectivity) $(L^m\mathrm{Uc}K)_n = 0$ if $m > n$, there is a largest m for which there exists $b \in (L^m\mathrm{Uc}K)_n$ with $b \notin \bar\zeta A$. By the density of $\bar\zeta A$, there exists $a \in \bar\zeta A \cap (L^m\mathrm{Uc}K)_n$ such that $a^m = b^m$. But then $b - a \in (L^{m+1}\mathrm{Uc}K)_n$, a contradiction. Q.E.D.

Given a Γ-bialgebra A with associated canonical map $\zeta\colon A \to Q_\Gamma A$, we write

$$P_\Gamma(A^0) = \{ f \in A^* \mid f(\mathrm{Ker}\,\zeta) = 0 \},$$

that is, $P_\Gamma(A^0) = \zeta^*(Q_\Gamma A)^*$. We henceforth regard ζ^* as a map of $(Q_\Gamma A)^*$ to $P_\Gamma(A^0)$, that is, we restrict the codomain to the image.

LEMMA 5.5. *$P_\Gamma(A^0)$ is a Lie subalgebra of $P(A^0)$ (with $P_\Gamma(A^0) = P(A^0)$ at characteristic* 0), *and $\zeta^*\colon (Q_\Gamma A)^* \to P_\Gamma(A^0)$ is a Lie algebra isomorphism.*

Proof. If $f \in P_\Gamma(A^0)$ then $f(ab) = \varepsilon(a) f(b) + f(a)\,\varepsilon(b)$ for all $a, b \in A$

since $f(k+(A^+)^2)=0$. Therefore $f\in A^0$ and $\Delta f=\varepsilon\otimes f+f\otimes\varepsilon$, that is, $f\in P(A^0)$. Since (A,ζ) is a cover, ζ^* is an injective Lie algebra map. Q.E.D.

We now give an interpretation of the map $\bar{\zeta}$ in terms of the Lie algebra $L=P_\Gamma(A^0)$ and an evaluation map on part of A^0. We let $\theta\colon UL\to A^0$ denote the canonical algebra map with $\theta\eta x=x$ for $x\in L$. Thus $\theta(UL)$ is the subalgebra of A^0 generated by L, and is a sub Hopf algebra of A^0. We let $\rho\colon A\to(UL)^0$ denote the map defined by $(\rho a)z=(\theta z)a$ for $a\in A$, $z\in UL$. Finally, we let $\lambda\colon K=Q_\Gamma A\to K^{*0}$ denote the canonical injective Lie coalgebra map given by Lemma 4.14.

THEOREM 5.6. *Let A be as in Theorem 5.3. Then for the canonical evaluation map $\rho\colon A\to(UP_\Gamma(A^0))^0$ (defined above) we have*

$$\rho=(\overline{\eta^0})^{-1}(\mathrm{Uc}(\zeta^{*0}))^{-1}\,\mathrm{Uc}(\lambda)\,\bar{\zeta}.$$

Moreover ρA is dense in $U(P_\Gamma(A^0))^*$ in the finite-open topology, and the canonical map $\theta\colon UP_\Gamma(A^0)\to A^0$ is injective.

Proof. We write $Q_\Gamma A=K$, $P_\Gamma(A^0)=L=K^*$. We want to show that $\mathrm{Uc}(\zeta^{*0})\,\overline{\eta^0}\rho=\mathrm{Uc}(\lambda)\,\bar{\zeta}$. Note that the image of each side is in $\mathrm{Uc}(K^{*0})$. Suppose $a\in A$ and $w_1,\dots,w_n\in K^*$. Then

$$\begin{aligned}(\mathrm{Uc}(\zeta^{*0})\,\overline{\eta^0}\rho a)^n(w_1\otimes\cdots\otimes w_n)&=(\overline{\eta^0}\rho a)^n(\zeta^*w_1\otimes\cdots\otimes\zeta^*w_n)\\&=\rho a((\eta\zeta^*w_1)\cdots(\eta\zeta^*w_n))=\theta((\eta\zeta^*w_1)\cdots(\eta\zeta^*w_n))\,a\\&=((\zeta^*w_1)*\cdots*(\zeta^*w_n))\,a=(\zeta^*w_1\otimes\cdots\otimes\zeta^*w_n)\,\Delta_{n-1}a\\&=((\otimes^n(\lambda\zeta))\,\Delta_{n-1}a)(w_1\otimes\cdots\otimes w_n)\\&=(\mathrm{Uc}(\lambda)\,\bar{\zeta}a)^n(w_1\otimes\cdots\otimes w_n)\end{aligned}$$

as desired. Next, to show that ρA is dense in $(UL)^*$ it suffices to show that if $0\neq z\in UL$ then there exists $a\in A$ such that $(\rho a)z\neq 0$; since $(\rho a)z=(\theta z)a$, this will also show that θ is injective. Taking a basis of standard monomials of UL we may write $z=y+\mathrm{LT}$ (LT denotes lower terms) where

$$y=\sum_{e_1+\cdots+e_n=t}\alpha_{e_1,\dots,e_n}(\eta x_1)^{e_1}\cdots(\eta x_n)^{e_n}$$

with $x_1,\dots,x_n$ linearly independent in $L=K^*$. Since $\bigcap_{i=1}^n\operatorname{Ker}x_i$ has finite codimension in K, there exist $v_1,\dots,v_n\in K$ such that $x_iv_j=\delta_{ij}$ $(i,j=1,\dots,n)$. Suppose for some $(m_1,\dots,m_n)$, with $m_1+\cdots+m_n=t$, that $\alpha_{m_1,\dots,m_n}=\alpha\neq 0$. By the density in Theorem 5.3, there exists $a\in A$ such that $(\bar{\zeta}a)^i=0$ for $i\leqslant t-1$ and

$$(\bar{\zeta}a)^t=(\gamma_{m_1}v_1)\cdot\cdots\cdot(\gamma_{m_n}v_n).$$

Then with $x_i = \zeta^* w_i$ we have, as above,

$$\begin{aligned}(\rho a)\, z &= (\rho a)(y + LT) = (\overline{\eta^0} \rho a)^t \Big(\sum \alpha_{e_1,\dots,e_n}(\otimes^{e_1} x_1) \otimes \cdots \\ &\quad \otimes (\otimes^{e_n} x_n)\Big) + \mathrm{LT}' \\ &= (\mathrm{Uc}(\zeta^{*0})\, \overline{\eta^0} \rho a)^t \Big(\sum \alpha_{e_1,\dots,e_n}(\otimes^{e_1} w_1) \otimes \cdots \\ &\quad \otimes (\otimes^{e_n} w_n)\Big) + \mathrm{LT}'' \\ &= (\mathrm{Uc}(\lambda)\, \zeta a)^t \Big(\sum \alpha_{e_1,\dots,e_n}(\otimes^{e_1} w_1) \otimes \cdots \otimes (\otimes^{e_n} w_n)\Big) + 0 \\ &= \Big(\sum \alpha_{e_1,\dots,e_n}(\otimes^{e_1} w_1) \otimes \cdots \otimes (\otimes^{e_n} w_n)\Big)\, ((\gamma_{e_1} v_1) \cdot \cdots \\ &\quad \cdot (\gamma_{e_n} v_n)) = \alpha\end{aligned}$$

since $(\otimes^{e_i} w_i)(\gamma_{e_j} v_j) = \delta_{ij}$. Q.E.D.

Note that if $Q_\Gamma A$ is infinite dimensional then $\mathrm{Uc}(\lambda)$ need not be surjective. Thus the image of ζ is dense in a space which can correspond to a proper subspace of $U(P_\Gamma(A^0))^0$. In this sense ζ gives more information than ρ does.

COROLLARY 5.7. *$\{a \in A \mid$ the subalgebra of A^0 generated by $P_\Gamma(A^0)$ annihilates $a\} = \mathrm{Ker}\ \rho = \gamma^\infty A$.*

We recall [Sw] that an algebra B is called *proper* if B^0 separates points of B, or equivalently, if the intersection of all cofinite ideals of B is 0.

COROLLARY 5.8. *For any vector space V, $\mathrm{Tc}V$ (and hence any subalgebra of $\mathrm{Tc}V$) is a proper algebra. In particular, if A is as above and $\gamma^\infty A = 0$ then A is a proper algebra.*

Proof. $\gamma^\infty \mathrm{Tc}V = 0$, and so the result for $\mathrm{Tc}V$ follows from Corollary 5.7. Q.E.D.

APPENDIX: CHARACTERISTIC 2 WITH ODD ELEMENTS PRESENT

Throughout this appendix we assume that the characteristic is 2. We modify several of the definitions of Section 4 in order that the results of

Sections 4 and 5 remain valid when nonzero elements of odd degree are allowed.

For any vector space W we write

$$(W\otimes W)' = W\otimes W/(1+\tau)\, W\otimes W$$

and define a linear map $\sigma=\sigma_W$ of $W\otimes W$ to $(W\otimes W)'$ by setting, for $v\in W_i$, $w\in W_j$, $\sigma(v\otimes w)=0$ unless $i=j$ is odd, in which case

$$\sigma(v\otimes w)=v\otimes w+(1+\tau)\, W\otimes W.$$

If $\varphi\colon V\to W$ is linear then there corresponds to $\sigma(\varphi\otimes\varphi)$ a linear map, which we denote by $(\varphi\otimes\varphi)'$, of $(V\otimes V)'$ to $(W\otimes W)'$. If C is a coalgebra then $C^{[-]}$ has, aside from $\delta=(1+\tau)\,\Delta$, an extra operation which we call $\mu=\mu_C$, given by $\mu_C=\sigma\Delta\colon C\to(C\otimes C)'$. We now define a Lie coalgebra (in the new sense) to be a vector space K with linear maps $\delta\colon K\to K\otimes K$ and $\mu=\mu_K\colon K\to(K\otimes K)'$ such that there is a cover (C,ω), that is, a coalgebra C and surjective linear map $\omega\colon C\to K$ satisfying $\delta\omega=(\omega\otimes\omega)(1+\tau)\,\Delta$ (as before) and

$$\mu_K\omega=(\omega\otimes\omega)'\,\mu_C \qquad (=\sigma(\omega\otimes\omega)\,\Delta_C).$$

Note that $\mu K_n=0$ unless $n=2i$ for odd i. We also require Lie coalgebra maps to respect not only δ but also μ, that is, if $\varphi\colon K\to M$ then $\delta_M\varphi=(\varphi\otimes\varphi)\,\delta_K$ and $\mu_M\varphi=(\varphi\otimes\varphi)'\,\mu_K$.

We modify the definition of the Lie property of $a\in \mathrm{Tc}K$ by requiring, in addition to $\delta a^1=(1+\tau)\,a^2$, that $\mu a^1=\sigma a^2$. Also we add to the definition of $a\in\mathrm{Tc}K$ being Lie-symmetric the condition that

$$(1^j\otimes\mu\otimes 1^{n-1-j})\,a^n=(1^j\otimes\sigma\otimes 1^{n-1-j})\,a^{n+1}$$

for $0\leqslant j\leqslant n-1$ and $n\geqslant 1$. We call $a\in\mathrm{Tc}V$ symmetric if it is Lie-symmetric when δ and μ are taken to be 0, that is, we require, for $n\geqslant 2$, in addition to the usual condition for symmetry, that a^n satisfy $(\sigma\otimes 1^{n-2})\,a^n=0$ (or equivalently $(1^j\otimes\sigma\otimes 1^{n-j-2})\,a^n=0$ for $0\leqslant j\leqslant n-2$). Thus if $n=2$ the former notion of symmetry required that $a^2\in\mathrm{Ker}(1+\tau)$ while the new notion also requires $a^2_{2i}\in\mathrm{Im}(1+\tau)$ for all odd i. Note that $\mathrm{Im}(1+\tau)\subseteq\mathrm{Ker}(1+\tau)$. The new requirement thus says that a^2_{2i} has no nonzero diagonal term $\alpha(x\otimes x)$.

Similarly, we modify the definition of cocommutativity of a coalgebra C to require for all $c\in C$ that in addition to $\Delta c\in\mathrm{Ker}(1+\tau)$ we have $\Delta c_{2i}\in\mathrm{Im}(1+\tau)$ (and thus Δc_{2i} has no diagonal term) for all odd i. Finally, in the definition of an abelian Lie coalgebra $[V]$ we require that $\mu=0$ (as well as $\delta=0$).

The notion of a Lie coalgebra under the old definition can be included under the new definition by doubling degrees, and similarly for the other modified concepts. (Thus in modifying the definitions we lose no generality, and avoid words such as "strictly" and "adjusted.")

With our modified definitions, all the results of Section 4 through Theorem 4.9 and of Section 5 through Corollary 5.4 go through as stated, with minor additions to the proofs. We give one sample of the change required, leaving the rest of the modifications of the proofs to the reader. In the proof of Proposition 4.1, we wish to show that $\bar{\varphi}D \subseteq \mathrm{Uc}K$, that is, assuming that φ respects μ, we must show that elements of $\bar{\varphi}D$ have the Lie property. We have

$$\mu\pi\bar{\varphi} = \mu\varphi = (\varphi \otimes \varphi)' \mu_D = \sigma(\varphi \otimes \varphi) \Delta$$
$$= \sigma(\pi \otimes \pi)(\bar{\varphi} \otimes \bar{\varphi}) \Delta = \sigma(\pi \otimes \pi) \Delta\bar{\varphi}$$

which gives the result.

We now comment on the relationship between Lie algebras and Lie coalgebras in our new sense, extending the remaining parts of Sections 4 and 5 to the present case. In [Sj] Sjödin defines an adjusted Lie algebra to be a Lie algebra L with an extra operation κ, defined on $\bigcup_{i \text{ odd}} L_i$ and sending L_i to L_{2i}, for which there is an envelope (A, θ) of L such that $\theta\kappa x = (\theta x)^2$ for all $x \in L_i$ and all odd i. He then defines the adjusted enveloping algebra of L, obtained by adjoining to the usual relations on TL the relations $x^2 - \kappa x$ ($x \in L_i$, i odd). This algebra he denotes by WL, but it will be denoted here by UL.

Now if K is a Lie coalgebra (in our new sense) then it can be proved that K^* is an adjusted Lie algebra, with $\kappa x = (x \otimes x)' \mu$. In the other direction, if L is adjusted and UL is its adjusted enveloping algebra, with canonical injection denoted by η, then again we take $L^0 = \eta^*((UL)^0)$. With these changes it can be proved that the rest of Sections 4 and 5 goes through; in particular, L^0 is a Lie coalgebra (in the new sense) and $\mathrm{Uc}(L^0) \cong (UL)^0$ as Hopf algebras, for L an adjusted Lie algebra.

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